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# Credit Risk Measurement With Wrong Way Risk

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## Abstract
I will start with introducing the corporate bond and several important components of it. The existing credit risk model can be categorized into two groups — Structural (Firm Value) Model and Reduced-form (Intensity-based) Models, followed by the risk measure and the risk measure—Value at Risk and its computation. Then I applied the previously introduced material to the given portfolio to calculate its credit VaR using two methods, S-critical and the Monte Carlo simulation. Finally, I present some advanced credit risk models with stochastic interest rate.

**Key words:** Credit risk measurement; Wrong way risk

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## 1. INTRODUCTION

According to Bielecki and Rutkowski (2002) the definition of credit risk is given as the risk caused by any events related to credit. For example, changes in the credit rating and default events may lead to fluctuation in financial position. The actual loss depends on the amount at risk and the amount that recovered (Bielecki and Rutkowski, 2002).

In order to look deeper into credit risk, I introduce two kinds of credit risk. Firstly, reference (credit) risk is the risk caused by the reference entity (third party) that plays an important role in the contract settlement, while both parties of the given agreement are assumed to be default-free (Bielecki and Rutkowski, 2002). In order for the market participant to trade without this kind of risk, credit derivatives are introduced into the market. It is a financial instrument that transfers the reference risk between the counterparty. Secondly, the counterparty (credit) risk is that the counterparty of the agreement faces the default risk of other party (Segoviano and Singh, 2008). By carefully quantifying the counterparty credit risk, one can correctly measures the value of the contract such as vulnerable claims and defaultable swaps. The contract is said to be associated with one-side or both side's default risk, depends on whether we consider the default event of one or both of the participants.

Generally speaking, as stated by Jorion (2007), credit risk is determined by three risk factors:

• Default risk: the risk of financial loss caused by the failure of counterparty to meet the obligation stated in the contract and is evaluated by the Probability of Default (PD)

• Credit exposure risk: the risk of fluctuation of the spot price of the counterparty and is evaluated by Exposure at Default (EAD)

• Recovery risk: the risk on the amount recovered after default and is measured by Loss Given Default (LGD)

In the following section, I will introduce corporate bonds, vulnerable claims and credit derivatives, which are financial instruments that closely related to credit risk.

## 2. CORPORATE BONDS

By issuing corporate bonds, the corporate has the obligation to pay a specific payment to the bond holder at a given date (maturity), stated by U.S. Securities and Exchange Commission (U.S Securities and Exchange Commission). Nevertheless, the default of the company may occur before the agreed date of payment. Thus, it may fail to fulfil the condition of making the payment to the bond holder, who will endure the financial loss. Noted that the default event, usually caused by firm's bankruptcy, is significative only if it occurs during the lifetime of the bond, i.e. before the maturity.

Defaultable claims, such as defaultable bond (or risky bonds) is the claim with possibility to default( Bielecki, 2004). On the other hand, the default-free bond (or risk-free bond or treasury bond) pays definitely the coupon and the notional price to the buyer at the expiration. Certainly, any bond holder exposes to some kind of risk. Here risk-free refers to the risk of bonds with the highest credit quality (Bielecki and Rutkowski, 2002).

To be more specific, first of all, assume the notional price of the corporate bond is K units of cash, and the bond pays no coupon before the maturity time denoted by T. Then set the time t arbitrage price of T-maturity bond is DB (t, T). If the firm does not default before the maturity, then DB (T, T) = K. When K=1, let B (t, T) represent the time t price of a zero-coupon bond with maturity T, then obviously B (T, T) =1.

### 2.1 Recovery Rules

In general, recovery schemes stated the date and the payment to the bond holder given that the default event happens prior to the expiration date (Bielecki and Rutkowski, 2002). The amount of payment is normally defined by rate ϕ, implies that the underlying firm pays partial value of the bond to the counterparty given default. Moreover, the timing is also a significant factor. Usually the default firm makes the recovery payment either at the time when default occurs or at the previously specified date (Bielecki and Rutkowski, 2002).

If the payment is made according to rate ϕ and is made at the time of default, denoted as τ, then such scheme is called fractional recovery of par value (Bielecki and Rutkowski, 2002). Suppose that the price of the bond K=1, then the value of the T-bond at time t, denoted as $\tilde{B}^{\phi}(t,T)$ is:

$$\tilde{B}^{\varphi}(T,T) = \mathbf{1}_{\{\tau>T\}} + \varphi B^{-1}(\tau,T)\mathbf{1}_{\{\tau\le T\}} \quad (2.1)$$

On contrast, if the payment is made at maturity, such scheme is called the fractional recovery of treasury value (Bielecki and Rutkowski, 2002), and the payoff is

$$\tilde{B}^{\varphi}(T,T) = \mathbf{1}_{\{\tau>T\}} + \varphi \mathbf{1}_{\{\tau\le T\}} \quad (2.2)$$

In addition, instead of the notional value of bond at maturity, the fractional recovery of market value specifies that the holder receives partial pre-default price of the bond at time of default (Bielecki and Rutkowski, 2002). Therefore, the payoff at time T is:

$$\tilde{B}^{\varphi}(T,T) = \mathbf{1}_{\{\tau>T\}} + \varphi B(\tau-,T)B^{-1}(\tau,T)\mathbf{1}_{\{\tau\le T\}} \quad (2.3)$$

where B(τ-, T) represents the spot price of the bond just before the default.

As mentioned before, normally the loss given default (LGD) can be used to measure the likely loss given default, implies that LGD is 1 − ϕ. (Bielecki and Rutkowski, 2002)

### 2.2 Cross Default and Default Correlation

Cross default is defined as "A provision in a loan agreement or other debt obligation stating that the borrower defulats if he/she goes into default on any other obligation."[1].

Now consider two defaultable claims A and B, whose lifetime intersect. Then let X be an indicator function of the event "claim A default", i.e. X takes 1 if A default, and 0 otherwise. Similarly, Y can be defined as the indicator function of event "claim B default". According to David X. Li (1999), the default correlation between A and B is equivalent to the correlation coefficient between the random variable X and Y.

### 2.3 Vulnerable Claims

Vulnerable claims are the contingent agreement traded between parties that have the possibility to default. As a result, each party faces the risk that another participant may default, deduced that the default risk plays an important part in valuation of vulnerable claims (Bielecki and Rutkowski, 2002).

#### 2.3.1 Vulnerable Claims with One-side Default Risk

Stated by Johnson and Stulz(Johnson and Stulz, 1987), the payoff of Vulnerable European Option, which is one of the major examples of vulnerable claims with one-side(unilateral) default risk, is dependent of the default of the underlying party occurs by the maturity and is irrelevant to the default of the holder.

For example, for a non–defaultable U-maturity zero-coupon bond, with expiration T<U , the price of the bond is represented by ND(T, U) and the exercise price is L. Then payoff of the European call option $EC_T$ at exercise date T is

$$EC_T = (ND(T,U) - L)^+ \quad (2.4)$$

Let τ denotes the time of default of the call writer (τ ≤ T), and ϕ denotes the recovery rate. Then the payoff of the claim is:

$$EC_T^d = EC_T\mathbf{1}_{\{\tau>T\}} + \varphi EC_T\mathbf{1}_{\{\tau\le T\}} \quad (2.5)$$

Generally, for a defaultable bond with price DB(T, U) and strike price L, the value of the call option can be interpreted as

$$EC_T = (DB(T,U) - L)^+ \quad (2.6)$$

Similarly, the payoff of the claim is

[1] http://financial-dictionary.thefreedictionary.com/Cross-Default

$$EC_T^d = EC_T \mathbf{1}_{\{\tau>T\}} + \varphi EC_T \mathbf{1}_{\{\tau\leq T\}}$$

(2.7) (Bielecki and Rutkowski, 2002)
(Johnson and Stulz, 1987)

### 2.3.2 Vulnerable Claims with Two-side Default Risk

Vulnerable claims with two-side(bilateral) counterparty risk are the contracts that both parties are exposed to counterparty risk, stated by Bielecki and Rutkowski (2002). The prime example is defaultable swap, which is a swap agreement between two defaultable parties. Comparing to default-free swap, the recovery rules at the default time will greatly affect the valuation of defaultable swaps (Bielecki and Rutkowski, 2002).

## 2.4 Credit Derivatives

Credit derivatives are the tool that is designed to transfer the risk related to credit to a third party, in order to prevent financial loss of the borrower or the lender. In this section, I will introduce the three prime examples: forward contract, default swaps and options(Hull, 1993).

### 2.4.1 Forward Contracts

Forward contract gives the obligation for the buyer to purchase a specific amount of some product at an agreed price at a future date. This kind of contract avoid the risk that the other might not fulfill the contract. For example, one can buy a forward contract for pound in order to eliminate the exposure to exchange rate risk(Hull, 1993).

### 2.4.2 Default Swaps and Default Options

Default swaps and options (or default insurance and protection) can be taken as insurance contract for the default event. According to Bielecki and Rutkowski (2002),

> "In these agreement, periodic fixed payment (for a default swap) or an upfront fee (for a default option) from the protection buyer is exchanged for the promise of some previously agreed payment from the protection seller to be made only if a particular credit event happens."

If the specified credit event happens before the contract expiration, the protection seller makes payment to the buyer to cover the loss. Otherwise, the obligations of both sides terminate at maturity.

Here are some most important components of default swaps/options (Bielecki and Rutkowski, 2002):

• The definition of "default": for example, it may include bankruptcy, payment default or downgrade of credit rating.

• The contingent default payment: it may affected by the price fluctuation of the underlying security or it can be a fixed amount stated at the contract.

• The specification of periodic payment: it is greatly affected by the credit quality of the security.

For standard default swaps or options, let T denotes the maturity time, and L represents the face value of a defaultable zero-coupon bond with expiration date $U \geq T$. The recovery payment is made at time of default τ. For a default option, the protection buyer makes the payment of a premium at the start of contract, and the price at τ is

$(L - D(\tau,U))^+ \mathbf{1}_{\{\tau\leq T\}}$ (2.8)

This kind of contract is defined as default put option (Bielecki and Rutkowski, 2002). Due to the upfront payment made by the long party, i.e. the protection buyer, the buyer's credit quality has no relevance in the contract.

For default swap, at times $T_i$, i=1,...,m before τ or expiration date given that default has not occurred, an annuity (credit swap premium) is paid by the protection buyer. In case of the buyer, the payoff of the default swap is:

$$L - D(\tau, U)\mathbf{1}_{\{\tau\leq T\}}\mathbf{1}_{\{\tau\}}(t) - \sum_{i=1}^{m} \kappa \mathbf{1}_{\{min(\tau,T)>T_i\}}\mathbf{1}_{\{T_i\}}(t) \qquad (2.9)$$

Where κ is the amount of annuity payment. (Bielecki and Rutkowski, 2002). The following graph shows the movement of credit risk between the protection buyer and protection seller of credit default swap.

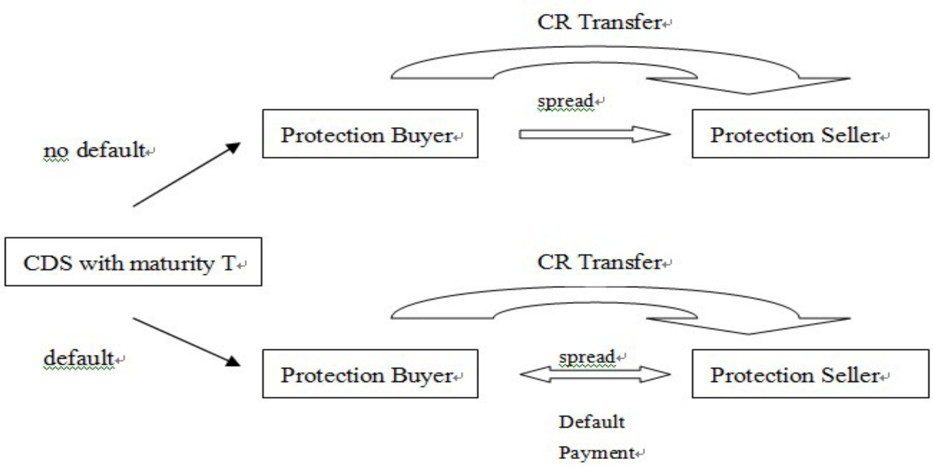


**Figure 1**
**Credit Default Swaps**

 

## 3. MODELS OF CREDIT RISK

### 3.1 Structural Models

For structural models, the major question is the valuation of capital structure, cited by Wang(2009)(Wang, 2006). In this case, the firm's liabilities are considered as contingent claims, compared with the value of firm's assets. As a result, the structural approach is also referred to as firm value approach or the option-theoretic approach (Bielecki and Rutkowski, 2002).

The default event is defined by the changes of total value of the firm's asset, denoted by V, and the default barrier. In other words, the firm's ability to fulfil the obligation stated on the contract is assumed to be solely depend on the value process $V_t$. In structural models, the default event is triggered if the firm's value is less than the default triggering barrier (Bielecki and Rutkowski, 2002).

One of the disadvantages of the structural approach is that it is designed under the assumption that the value of the firm can be directly determined. In addition, this model is valid provided that the firm's value is represented by securities with high liquidity(Wang, 2006).

#### 3.1.1 Defaultable Claims

Firstly, we specify a date $T^* > 0$, and assume that the probability space (Ω, F, P) is adapted to filtration F. Then I introduce the following notations:

- r: the interest rate process
- V: the firm's value process
- v: the barrier process, determining the time of default
- X: the promised contingent claim, which represents the firm's liabilities before T ≤ T∗
- A: promised dividends
- : the recovery claim, which is the recovery payment at the maturity T if default occurs by T
- Z: the recovery process, which is the payoff at default time given that the default occurs by maturity date
- P: practical probability measure
- P∗: the risk-neutral probability
- τ: τ=inf(t > 0 : Vt < vt)

If the default does not occur by expiration T, the agreed contingent claim X will be paid. Otherwise, the claim holder receive either $Z_\tau$ at time of default τ or at the expiration T (Bielecki and Rutkowski, 2002).

#### 3.1.2 Merton's Firm Value Model

The classic structural model—Merton's approach focuses on a firm with a single liability of delivering the previously agreed amount K. According to Jeanblanc (2006)( Bielecki, Jeanblanc and Rutkowski, 2006), following are some fundamental assumptions of this model:

- market participant can trade continuously
- All traded assets are infinitely dividable
- Trading of capital is at the identical interest rate r
- No limitation on the short-selling of assets
- No transaction loss and taxes
- The bankruptcy and reorganization costs in case of default are neglected

Suppose that the interest rate r is deterministic. As a result, the price at time t of the non–defaultable zero-coupon bond with expiration T and terminal payoff 1 is $B(t,T) = e^{-r(T-t)}$. Let $E(V_t)$ and $D(V_t)$ represent the amount of the firm's assets and liabilities at time t respectively, inferred that the firm value is $V_t = E(V_t) + D(V_t)$. Moreover, we suppose that under the spot martingale measure $P^*$, the firm's value process is driven by the geometric Brownian motion,

$$dV_t = V_t((r-\kappa)dt + \sigma dW_t^*)$$

where σ is the deterministic volatility. If the constant κ is non-negative then it represents the payout ratio, otherwise it stands for an inflow of money to the firm. $W^*$ is the one-dimensional Brownian motion under $P^*$, adapted to filtration F.

To be more specific, at the time of maturity, if the total value of the firm's asset $V_T$ falls below the face value of the liability, the firm defaults and the claim holder can get the amount $V_T$( Bielecki, Jeanblanc and Rutkowski, 2006). Otherwise, default event does not occur and the debt is paid in full. Hence it is a defaultable claim with recovery at maturity. In terms of the notation of defaultable claim, we can deduce that ( Bielecki, Jeanblanc and Rutkowski, 2006) $X = K, A = 0, = V_T, \tau = T\mathbf{1}_{\{VT<K\}} + \infty\mathbf{1}_{\{VT\geq L\}}$

The terminal payoff (firms debt) at maturity T is

$$^{1}(T) = K1_{\{\tau>T\}} + V_T 1_{\{\tau\leq T\}} \quad (3.1)$$

$$= K1_{\{V_T\geq K\}} + V_T 1_{\{V_T<K\}} \quad (3.2)$$

$$= \min\{V_T, K\} \quad (3.3)$$

$$= K - (K - V_T)^+ \quad (3.4)$$

As a result, the liability at time t is

$$(V_t) = KB(t,T) - P_t = D(t,T) \quad (3.5)$$

Where $P_t = B(t,T)(K - V_T)^+$, which is a put option known as put-to-default, and D(t, T) is the value of the defaultable claim. Consequently, the firm's equity is

$$E(V_T) = V_T - D(V_T) \quad (3.6)$$

$$= V_T - \min\{V_T, K\} \quad (3.7)$$

$$= (V_T - K)^+ \quad (3.8)$$

(Bielecki, Jeanblanc and Rutkowski, 2006) Thus, in this case, the firm's equity can be taken as a call option on the firm's assets with exercise price K. Using the Black-Scholes formula(Nielsen, 1992) for the price of put option gives

$$P_t = KB(t,T)N(-d_2(V_T, T-t)) - V_t N(-d_1(V_t, T-t)) \quad (3.9)$$

Where for every t [0, T]

$$d_{1,2}(V_t, T-t) = \frac{ln(\frac{V_t}{K}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

(3.10)

the price of the defaultable bond can be represented as

$$D(t,T) = V_t N(-d_1(V_t, T-t)) + KB(t,T)(1 - N(-d_2(V_t, T-t))) \quad (3.11)$$
$$= V_t N(-d_1(V_t, T-t)) + KB(t,T)(N(d_2(V_t, T-t))) \quad (3.12)$$

(Breccia, 2012)

### 3.2 Reduced-form Models

The reduced-form approach models the unpredictable random default time or other credit events, stated by S.Rachev(Rachev, 2009). While the default event in structural models is triggered when the firm value is smaller than the barrier, on contrast, the price of the company is not considered for the intensity-based model (Rachev, 2009). In addition, the structure of the liabilities is not important for this model.

First of all, set a probability space($\Omega$,F,$Q^*$) with a d-dimensional standard Brownian motion $W_t^*$, where t [0, $T^*$], for some time horizon $T^*$ and F is the filtration. Suppose the interest rate r is F–measurable.

The main problem is to construct the default time τ. We introduce a non-decreasing and F–measurable process Ψ, and a strictly positive random variable η with the cumulative distribution function $F$ : $F(x) = Q^*\{\eta \le x\}$ for$x\epsilon\mathbb{R}$. Therefore, time of default τ is defined as (Bielecki and Rutkowski, 2002)

$$\tau = inf\{t\epsilon[0,T^*] : \Psi_t \geq \eta\} \quad (3.13)$$

## 4. RISK MEASURE

### 4.1 Approaches to Risk Measurement

The existing methods to evaluate the risk can be classified into four groups: the notional-amount approach, factor-sensitivity measures, risk measures depend on the loss distribution and risk measures based on scenarios. The well-established criteria for evaluating a risk measure are the coherence, and such risk measure is called coherent risk measure. To be more specific, I will present an example of such measure, the expected shortfall.

#### 4.1.1 Notional-amount Approach

According to Mcneil, Frey and Embrechts(2005)(Mcneil, Frey, and Embrechts, 2005), it is the initial approach to risk measurement and it defines the risk as the summation of the face values of the assets in the portfolio, with each factors is of certain weights given by the risk of the respective asset.

The advantage of this approach is its simplicity. On the other hand, the approach does not distinguish between long and short position, i.e. the risk of long or put are considered to be the same. In addition, it does not reflect the benefit of diversified investment. Finally, there may be large differences between the notional amount and the economic value(Mcneil, Frey, and Embrechts, 2005).

#### 4.1.2 Factor-sensitivity Measures

It measures the risk by the predetermined changes in a risk factor and the changes in portfolio value, usually it takes the form of derivative (known as the "Greeks"). The main drawback is that it cannot measure the overall risk of a given position. To be more specific, it is not reasonable to add up the risk driven by different factors (Mcneil, Frey, and Embrechts, 2005).

#### 4.1.3 Risk Measures depend on Loss Distribution

It is the most developed measure which depends on the conditional or unconditional loss distribution of the portfolio over a given time interval (Mcneil, Frey, and Embrechts, 2005), e.g. Value at Risk (VaR).

Since loss is the most important factor of risk management, it is a very functional approach when making financial decision. Moreover, we can compare different portfolio easily by evaluating the loss.

On the contrary, due to the fact that most estimation for loss distribution are based on the historical data, if there is some big changes in the financial market, the estimation may be invalid. Even if the environment remains unchanged, it is very complicated to simulate the loss distribution accurately (Mcneil, Frey, and Embrechts, 2005).

#### 4.1.4 Scenario-based Risk Measures

In this approach, the risk is measured by considering changes in some risk factors and the formal description given by Mcneil, Frey and Embrechts (2005) (Mcneil, Frey, and Embrechts, 2005) is as followed:

Specify a set X = $\{x_1,...,x_n\}$ representing the factor that closely related to risk and a vector w = $(w_1,...w_n)$ denoting the weights with $w_i$ [0,1] for all i. For a portfolio of risky securities and with corresponding loss operator $l_{[t]}$, the risk is computed as

$$\varphi_{[X,w]} = max\{w_1 l_{[t]}(x_1),...,w_n l_{[t]}(x_n)\} \quad (4.1)$$

The core of this approach is to determine the appropriate set of scenarios and weights (Mcneil, Frey, and Embrechts, 2005).

### 4.2 Coherent Risk Measure

Risk measure is of great significance when making decision for investment. Therefore the choice of risk measure is very important. we first write down a list of criteria for a good risk measurement ψ (Haugh, 2010) (Jorion, 2007).

• Monotonicity: if $V_1 \le V_2$, then $\psi(V_1) \ge \psi(V_2)$. Namely the portfolio with lower return has greater risk.

• Translation invariance: $\psi(V + k) = \psi(V) - k$. Adding k units of cash to a portfolio should lower the risk by k

• Homogeneity: $\psi(bV) = b\psi(V)$. Increasing the size of the portfolio by b should scale its risk by the same scale

 

• Subadditivity: $\psi(V_1 + V_2) \leq \psi(V_1) + \psi(V_2)$. Merging portfolios cannot increase risk.

Risk measure satisfying the above condition is called coherent risk measure.

### 4.3 Expected Shortfall

For some increasing function T: R → R, the generalized inverse function $T^{-1}$ is defined as

$$T^{-1} = \inf\{x\epsilon\mathbb{R} : T(x) \geq y\}$$

Then the quantile function of F is

$$q_\alpha(F) = F^{-1}(\alpha) = \inf\{x\epsilon\mathbb{R} : F(x) \geq \alpha\}$$

For loss L with mean $E() < \infty$ and probability distribution $F_L$, the expected shortfall at confidence level α is

$$ES_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 q_u(F_L)du$$

where $q_u(F_L) = F_L^{-1}(u)$ is the quantile function of $F_L$. Furthermore, the expected shortfall is a coherent risk measure(Mcneil, Frey, and Embrechts, 2005).

## 5. VALUE AT RISK

Value at Risk (VaR) is a risk measure depending on current position. Since it describes the risk by a single, easy-to-understand number, VaR has become an important tool for conveying trading risk to shareholder and senior manager. It associates with two quantitative factors, time horizon and the confidence level. The formal definition of VaR given by Mcneil, Frey and Embrechts(2005)(Mcneil, Frey, and Embrechts, 2005) is as followed :

Given some confidence level α [0,1]. The VaR of the portfolio at the confidence level α is the minimum value l such that the probability of loss L exceeding l is smaller or equals to (1 - α). Mathematically,

$$VaR_\alpha = \inf\{l\epsilon\mathbb{R} : P(L > l) \leq 1-\alpha\} = \inf\{l\epsilon R : F(L(l)) \geq \alpha\}$$

In other words, VaR is the quantile of the loss distribution.

In addition, the expected shortfall is related to VaR as

$$ES_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(L)du$$

However, the VaR of confidence level α cannot tell how severe the losses can be with probability lower than 1 − α. Furthermore, the VaR does not satisfy the last condition of coherent measure, the subadditivity. To be more specific, the short option positions can create large losses with a low probability and hence have low VaR yet combine to create portfolios with larger VaR (Jorion, 2007). Thus it contradicts with our observation that diversification of investment is an effective way to manage risk. Specifically, if the data is fitted by normal distribution, VaR is coherent. By the Central Limit Theorem (Hazewinkel and Michiel, 2001), which stated that the sum of independent random variables converges to a normal distribution, we can deduce that VaR is a coherent risk measure at the highest level of financial institution (Jorion, 2007).

Basically, there are two approaches to compute VaR, either by considering the actual empirical distribution (nonparametric) or by adapting a parametric approximation (parametric). To be more specific, the former method derived VaR by sample quantile and the latter approximate it by standard deviation.

### 5.1 Nonparametric VaR

Firstly, introduce the following notation:

- $W_0$: the initial investment
- R: rate of return
- $W = W_0(1 + R)$
- μ: excepted return of R
- σ: volatility of R
- $W^* = W_0(1 + R^*)$: the lowest portfolio value at the given confidence level α

The relative VaR is defined as (Jorion, 2007)

$VaR(mean) = E(W) - W^* = -W_0(R^* - \mu)$

And the absolute VaR is (Jorion, 2007)

$VaR(zero) = W_0 - W^* = -W_0R^*$

If the time horizon is short, there may be little difference between E(W) and $W_0$, and both method will give similar results.

More generally, let the probability distribution of the future portfolio value be f(w), then

$$\alpha = \int_{W^*}^{\infty} f(w)dw$$

$$1-\alpha = \int_{-\infty}^{W^*} f(w)dw = Prob(w \leq W^*)$$

The number $W^*$ is also known as quantile of the distribution, which is the cutoff point with a fixed probability of being overstepped (Jorion, 2007).

### 5.2 Parametric VaR

If the probability distribution can be categorized into a parametric family, the VaR can be calculated using the standard deviation multiplying with a factor that computed by the confidence level. As stated by Jorion (2007), we use normal distribution $\mathbf{N}(\mu,\sigma^2)$ to fit the data, then set

$W^* = W_0(1 + R^*)$

Normally, $R^*$ is negative, so it can be written as $-|R^*|$. Let z be standard normal deviate, and

$$-z = \frac{-|R^*| - \mu}{\sigma} \quad (5.1)$$

Equivalently,

$$1-\alpha=\int_{-\infty}^{W^*} f(w)dw=\int_{-\infty}^{-\|R^*\|} f(r)dr=\int_{-\infty}^{-z} N(\varepsilon)d\varepsilon \qquad (5.2)$$

where N is the probability density function of standard normal distribution. By the inverse function of normal distribution, z can be computed. Then the cutoff return is given as

$R^* = -|R^*| = -z\sigma + \mu$ (5.3)

Provided that all the uncertainty is contained in σ, this method can be generalized to other distributions (Jorion, 2007).

### 5.3 Portfolio VaR

Computation for the VaR of financial instrument with single risk factor has been discussed previously. Now we consider more general case - the financial instrument with many risk factors, i.e. the portfolio risk. A portfolio can be described as a combination of certain number (N) of constituent assets. If there is no trading over the selected time horizon, i.e. the position remains the same, the portfolio rate of return is a linear combination of the rate of return($R_i$) of the underlying assets, where the weights($x_i$) are given as the percentage of initial investment to each asset (Kondapaneni, 2005).

$$R_P = x_1R_1 + x_2R_2 + ... + x_NR_N = \sum_{i=1}^{N} x_iR_i \qquad (5.4)$$

Then the excepted return of the portfolio is

$$E(R_P) = \mu_P = \sum_{i=1}^{N} x_iE(R_i) = \sum_{i=1}^{N} x_i\mu_i \qquad (5.5)$$

The variance is

$$Var(R_P) = \sigma_P^2 \qquad (5.6)$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} x_ix_j\sigma_{ij} \qquad (5.7)$$

$$= \sum_{i=1}^{N} x_i^2\sigma_i^2 + 2\sum_{i=1}^{N}\sum_{j<i} x_ix_j\sigma_{ij} \qquad (5.8)$$

If all individual securities are normally distributed, then the portfolio return, which is an linear combination of normal random variables, also follows normal distribution with mean $\mu_P$ and variance $\sigma_P^2$. Assume that the probability of observing a loss worse than -z is α, and let W denotes the initial portfolio value, then the VaR of the portfolio is

$VaR_P = z\sigma_P W$ (5.9)

For risk management, we can control the risk by lower correlations or larger amount of assets (Jorion, 2007).

### 5.4 Delta-Normal Valuation

Let $V_t$ be the value of assets and $S_t$ be the risk factor, and linear function V:R→R is a mapping from S to V, and

$V_0 = V(S_0)$

In addition, define $\Delta_t$ as the first partial derivative, which is equivalent to the changes in value subject to the fluctuation in prices, evaluated at the current position $V_0$. This is known as Δ for a derivative for a fixed-income portfolio. The potential loss in value is then calculated as

$$dV = \frac{\partial V}{\partial S}\|_0 dS = \Delta_0 dS \qquad (5.10)$$

where dS is the potential change in prices. Since $V_t$ and $S_t$ are linearly dependent, if it is normally distributed, the portfolio VaR can be computed as

$$R = \|\Delta_0\| VaR_S = \|\Delta_0\|(z\sigma S_0) \qquad (5.11)$$

where $VaR_S$ is the VaR for the underlying risk factor, z is the standard normal deviate for a given confidence level, and σ is the variance of the given portfolio (Jorion, 2007).

 

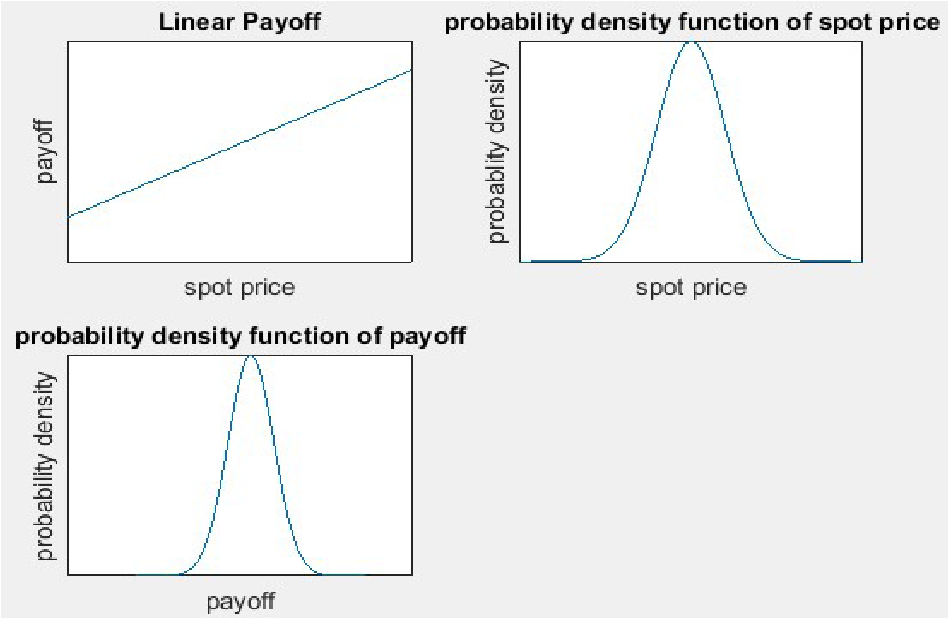


**Figure 2**
**Distribution with linear exposures**

### 5.5 Delta-Gamma Approximation

Delta-normal approach is not applicative in the situation when the value and the risk factor are non-linearly dependent. For example, consider a long position in a call option. Although it is not linearly dependent, but we can still describe the relation easily since it's a monotonic transformation.

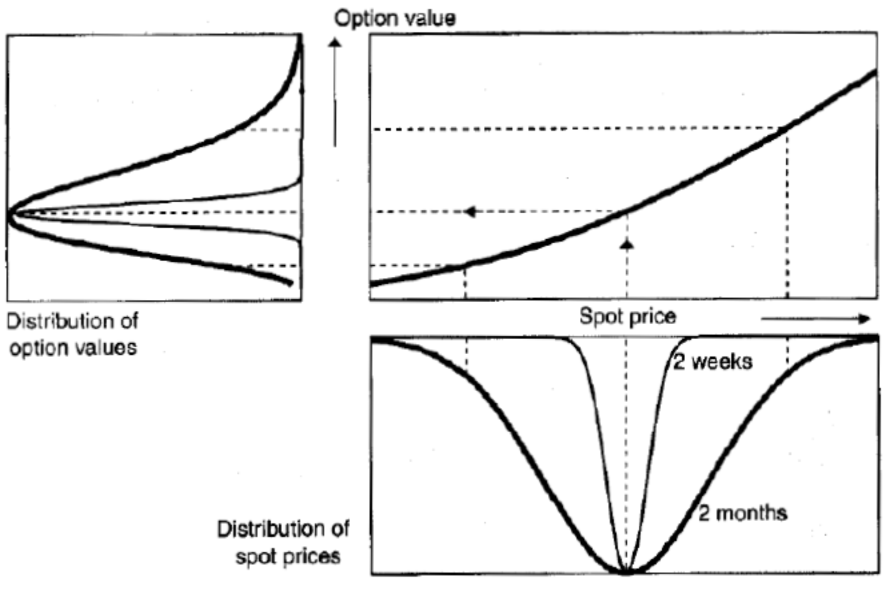


**Figure 3**
**Transformation of distribution (Jorion, 2007)**

The above graph shows how the risk factor distribution can be transformed into the distribution of the option value. The c quantile of V can be calculated by the c quantile of S. The worst loss for specified confidence level is $S^* = S_0 - \alpha\sigma S_0$. Due to the monotonicity of the mapping function V, VaR can be computed as

$VaR = V(S_0) - V(S_0 - \alpha\sigma S_0)$ (5.12)

Now consider the dynamics of valuation function, using Taylors expansion(Weisstein, 2004), we can extend the above delta-normal method with higher order terms

$$dV = \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 + \frac{\partial V}{\partial t}dt + \ldots$$

(5. 13)

 

$$= \Delta dS + \frac{1}{2}\Gamma dS^2 + \Theta dt + \ldots$$
(5.14)

where Δ and Γ are the first and second derivatives of the portfolio value respectively, and Θ is deterministic time drift. Then the VaR can be represented as (Jorion, 2007)

$$VaR = V(S_0) - V(S_0 - \alpha\sigma S_0)$$
(5.15)

$$= V(S_0) - [V(S_0) + \Delta(-\alpha\sigma S) + \frac{1}{2}\Gamma(-\alpha\sigma S)^2]$$
(5.16)

$$= \|\Delta\|(\alpha\sigma S) - \frac{1}{2}\Gamma(\alpha\sigma S)^2$$
(5.17)

More generally, this formula is applicable to long and short positions in call and put options, and all other valuation function V that is monotonic to S (Jorion, 2007).

Nevertheless, this method cannot be applied to the situation when the payoff and the price are non-monotonic. To be more specific, one of the example is the long straddle, which is a strategy to purchase the call and put option on the same equity with identical expiration and exercise price (Jorion, 2007). Consequently, the trader will suffer the greatest loss if the price remains the same, demonstrating by the following graph:

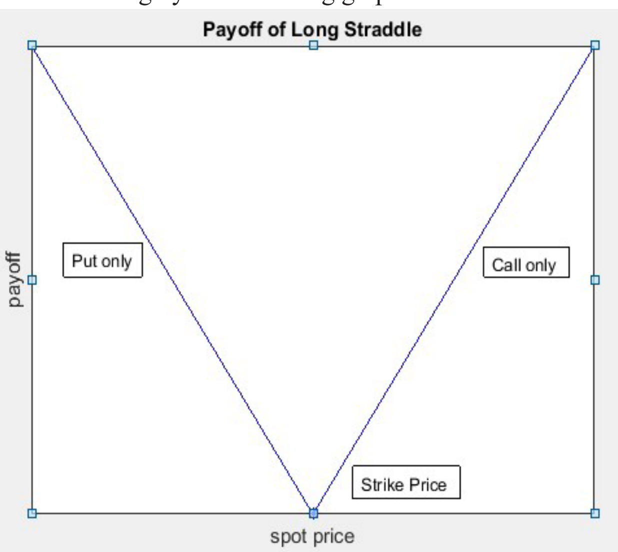


**Figure 4**
**Long Straddle**

In this case, it is inadequate to evaluate the price at the extreme value. Thus we need to use the full-valuation approach which considers all the intermediate value and the profit or loss (negative profit) is given as (Jorion, 2007)

$dV = V(S_T) - V(S_0)$ (5.18)

Where the price $S_T$ can be simulated by the Monte Carlo method. This method is potentially more accurate and the VaR is computed as the percentiles of the loss.

### 5.6 Monte Carlo Method

Due to its flexibility, monte carlo simulation is widely used in financial institutions to value complex derivatives and measure risk. However, this approach requires costly investment in intellectual and systems development (Kondapaneni, 2005).

> "The basic concept behind the Monte Carlo approach is to simulate repeatedly a random process for the financial variable of interest covering a wide range of possible situations." (Jorion, 2007)

These variables are driven by the probability distribution under our assumption. Hence by repeating the simulation, we can recreate the entire distribution of portfolio values, which leads to VaR.

#### 5.6.1 Simulating a Price Path

In this section I focus on the simplest case with only one random variable. Geometric Brownian Motion (GBM) model is one of the most widely-used models for option pricing. This model supposes that the price has independent increment in different time interval and (Jorion, 2007)

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t \qquad (5.19)$$

where $W_t$ is a brownian motion. The process is geometric since all parameters are multiplication of current price $S_t$. The parameters $\mu_t$ and $\sigma_t$ are the instantaneous drift and volatility respectively, which we assume to be deterministic constants.

Firstly, we introduce this method by a simple simulation. Practically, we can divide the path into lots of small time interval, then the process can be computed as discrete moves of size Δt. Let t and T be the present time and the expiration time respectively, and set $\tau = T - t$ as the VaR time horizon. To generate series of simulation of $S_t$, we divided τ into n intervals, with Δt=. Integrating over a finite interval gives

$$\Delta S_t = S_{t-1}(\mu\Delta t + \sigma\epsilon\sqrt{\Delta t}) \qquad (5.20)$$

where  is a standard normal random variable with mean 0 and variance 1. This process has mean E() =t, and variance Var()=. For simulation of the price path, starting from $S_t$ and generate a sequence of () for i = 1,2,...,n. Then

$$S_{t+1} = S_t + S_t(\mu\Delta t + \sigma\epsilon_1\sqrt{\Delta t}) \qquad (5.21)$$

And $S_{t+2}$ can be computed similarly. Generally,

$$S_{t+i+1} = S_{t+i} + S_{t+i}(\mu\Delta t + \sigma\epsilon_i\sqrt{\Delta t}) \qquad (5.22)$$

For $0 \le i \le n - 1$ and $T = t + n$.

Now we adapt a more complex and accurate simulation given by J.Armstrong(2015)(Armstrong, 2015), which I will use for the calculation for Part 2. Solving the stochastic differentiate equation (SDE) given by the assumption

 

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t \qquad (5.23)$$

Gives

$$S_t = S_0 e^{\sigma_t W_t + (\mu_t - \frac{1}{2}\sigma_t^2)t} \qquad (5.24)$$

$$log(S_t) = log(S_0) + (\mu_t - \frac{1}{2}\sigma_t^2)t + \sigma_t W_t \qquad (5.25)$$

where $log(S_0)$ is a deterministic constant. Assume

$$s_t = log(S_t) = log(S_0) + (\mu_t - \frac{1}{2}\sigma_t^2)t + \sigma_t W_t \qquad (5.26)$$

$$s_0 = log(S_0) \qquad (5.27)$$

Therefore, we can deduce that $s_t$ is a Geometric Brownian motion which is normally distributed with mean and variance $\sigma_t^2 t$.

Proof: As t is deterministic quantitive factor and $W_t$ is standard Brownian motion.

$$E(s_t) = E[(\mu_t - \frac{\sigma_t^2}{2})t] + E(\sigma_t W_t) + E(s_0) \qquad (5.28)$$

$$= (\mu_t - \frac{\sigma_t^2}{2})t + s_0 \qquad (5.29)$$

$$Var(s_t) = E[(s_t - E(s_t))^2] \qquad (5.30)$$

$$= E[(\sigma_t W_t)^2] \qquad (5.31)$$

$$= \sigma_t^2 t \qquad (5.32)$$

Then we get,

$$ds_t = (\mu_t - \frac{1}{2}\sigma_t^2)dt + \sigma_t dW_t$$

(5.33)

For small time interval,

$$ds_t \to s_{t+1} - s_t$$

$$dt \to \Delta t$$

$$dW_t \to \sqrt{(\Delta t)}\epsilon$$

where the last approximation is deduced by Central Limit Theorem(Hazewinkel and Michiel, 2001). Then

$$s_{t+1} = s_t + (\mu_t - \frac{1}{2}\sigma_t^2)\Delta t + \sigma_t \sqrt{\Delta t}\epsilon$$

(5.34)

Then we can get the price path simulation of $S_t$

$S_t = e^{st}$ (5.35)

where  is a standard normal deviate. Therefore, to simulate the price path, we only need to generate a sequence of standard normal deviate and then substitute into the above equation to get the stock price for each step(Armstrong, 2015).

**5.6.2 Creating Random Numbers**

To generate a normally distributed deviate, as stated by Kyng and Konstandatos (2014) (Kyng and Konstandatos, 2014), firstly, we use the uniform distribution over the interval [0, 1], to generate a random variable x. Then transform the random number x into the desired distribution by the inverse of cumulative probability distribution function. By definition, the cumulative function N(y) lies in the interval [0, 1] which coincides with the interval of random number x. For example, to generate a normally distributed random variables y, we set $x = N(y)$, then $y = N^{-1}(x)$.

**5.6.3 Computation of VaR**

Given the simulation of price path, the VaR can be computed as follow (Jorion, 2007):

- Determine the stochastic price process and its parameters
- Generate a sequence of random variables, and then compute the sequence of price $S_t$
- Compute the value of the portfolio $V_{t+n} = V_T$ using the sequence of price
- Repeat the previous two steps as many times as possible, e.g. set K=10000000
- Compute the mean $E(V_T)$ and the quantile Q $(V_T, \alpha)$ of these K observations of $V_T$, and VaR is given by

$VaR(c,T) = E(V_T) - Q(V_T, \alpha)$ (5.36)

Alternatively, after simulating the price path, instead of the value (V), we can compute the loss using the price path directly. In this case, the VaR is the quantile of the loss computed by these K observation(Crouhy, Galai and Mark, 2000). The following graph explains the above procedure pictorially.

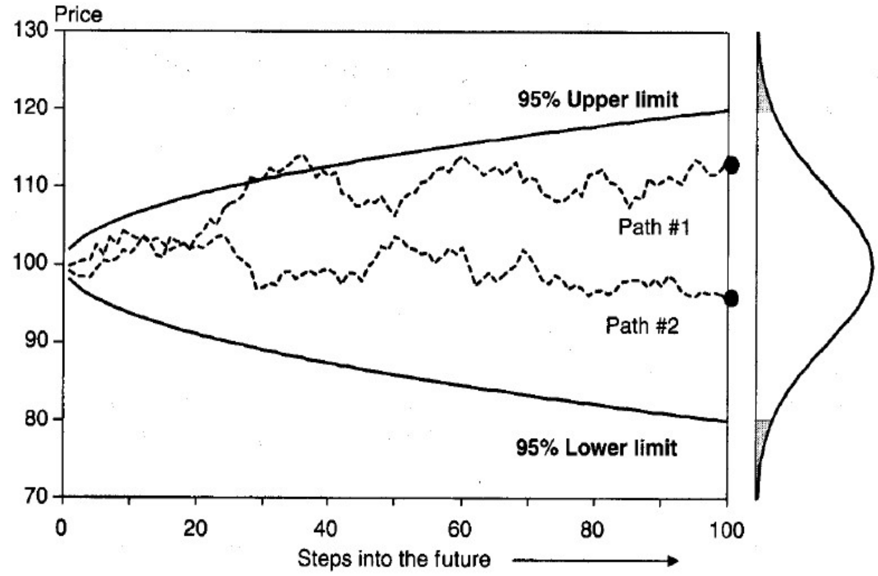


**Figure 5**
**Monte Carlo Simulation (Jorion, 2007)**

## 6. ANALYTICAL AND NUMERICAL APPLICATION

I use two methods to calculate the credit VaR of the given portfolio that long two call options and short one put option on the same equity stock. The first method is S-critical and the second is Monte Carlo Simulation. The former method is much quicker and can avoid the simulation, but it can only be applied to special cases for ρ=-1 or 0 or 1. The latter method is the most powerful among the existing simulation tools and it can be applied to general case for any ρ between -1 to 1, while it is very time consuming for very large amount of simulation which determines the accuracy.

### 6.1 Method 1(find the critical value of $S_t$)

Generally speaking, I compute the quantile of the underlying stock price $S_{crit}$ first, using the correlation between $S_t$ and $V_t$, and then substitute $S_{crit}$ into the function that maps from $S_t$ to portfolio value, which leads to loss and VaR.

**6.1.1 ρ = 0**

By definition (Jorion, 2007),

$$VaR_\alpha = inf\{l\epsilon\mathbb{R} : P(L > l) \leq 1 - \alpha\}$$
$$= \{l\epsilon\mathbb{R} : P(L > l) = 1 - \alpha\}$$

(6.1) and (6.2)

where L represent the loss. When the correlation coefficient between the Brownian motion driving underlying stock price and counterparty firm value is zero, the random variable(r.v) portfolio value P and counterparty firm value $V_t$ are independent. Therefore,

$$P(V_{1y} < D, P^+_{1y} > VaR_\alpha) = \alpha \quad (6.3)$$
$$P(V_{1y} < D)P(P^+_{1y} > VaR_\alpha) = \alpha \quad (6.4)$$
$$P(P^+_{1y} > VaR_\alpha) = \frac{\alpha}{P(V_{1y} < D)} \quad (6.5)$$

The dynamics of the counterparty firm value under physical measure P is given as

$$dV_t = \mu_v V_t dt + \sigma_v V_t d\bar{Z}_t \quad (6.6)$$

$V(0) = 1$, $\mu_v = 0.1, \sigma_v = 0.4005$ (6.7)

where  is the Brownian motion under P. Solving the above stochastic differential equation (SDE) gives

$$V_t = exp[(\mu_v - \frac{\sigma_v^2}{2})t + \sigma_v \bar{Z}_t]$$

(6.8)

As a result, the probability of counterparty default is

 

$$P(V_{1y} < D) = P(exp[(\mu_v - \frac{\sigma_v^2}{2})t + \sigma_v \bar{Z}_t] < D) \quad (6.9)$$

$$= P(\bar{Z}_t < \frac{lnD - (\mu_v - \frac{\sigma_v^2}{2})t}{\sigma_v}) \quad (6.10)$$

$$= P(\frac{\bar{Z}_t}{\sqrt{t}} < \frac{lnD - (\mu_v - \frac{\sigma_v^2}{2})t}{\sigma_v \sqrt{t}}) \quad (6.11)$$

$$= N(\frac{lnD - (\mu_v - \frac{\sigma_v^2}{2})t}{\sigma_v \sqrt{t}}) \quad (6.12)$$

Similarly, solving the SDE for underlying stock price

$$dS_t = \mu_s S_t dt + \sigma_s S_t d\bar{W}_t$$

$$S_0 = 100, \mu_s = 0.08, \sigma_s = 0.2$$

gives

$$S_t = 100exp[(\mu_s - \frac{\sigma_s^2}{2})t + \sigma_s \bar{W}_t] \quad (6.13)$$

Let the mapping function from price $S_t$ to portfolio value $P_t$ be
$g : R \to R$, $P_t = g(S_t) = 2BSC(S_t) - BSP(S_t)$
where BSC and BSP are the price of the Call and Put option respectively, using the Black Scholes Formula(Nielsen, 1992) gives

$$P(P_{1y} > VaR_\alpha) = P(g(S_{1y}) > VaR_\alpha) \quad (6.14)$$

$$= P(S_{1y} > g^{-1}(VaR_\alpha) = S_{crit}) \quad (6.15)$$

$$= P(lnS_{1y} > lnS_{crit}) \quad (6.16)$$

$$= P(lnS_0 + (\mu_s - \frac{\sigma_s^2}{2})t + \sigma_s \bar{W}_t > lnS_{crit}) \quad (6.17)$$

$$= P(\bar{W}_t > \frac{lnS_{crit} - lnS_0 - (\mu_s - \frac{\sigma_s^2}{2})t}{\sigma_s}) \quad (6.18)$$

$$= P(\frac{\bar{W}_t}{\sqrt{t}} > \frac{lnS_{crit} - lnS_0 - (\mu_s - \frac{\sigma_s^2}{2})t}{\sigma_s \sqrt{t}}) \quad (6.19)$$

$$= 1 - N(\frac{lnS_{crit} - lnS_0 - (\mu_s - \frac{\sigma_s^2}{2})t}{\sigma_s \sqrt{t}}) \quad (6.20)$$

$$= \frac{\alpha}{P(V_{1y} < D)} \quad (6.21)$$

which gives an equation

$$1 - N(\frac{lnS_{crit} - lnS_0 - (\mu_s - \frac{\sigma_s^2}{2})t}{\sigma_s \sqrt{t}}) = \frac{\alpha}{P(V_{1y} < D)} \quad (6.22)$$

$$lnS_{crit} = \sigma_s \sqrt{t} N^{-1}(1 - \frac{\alpha}{P(V_{1y} < D)}) + lnS_0 + (\mu_s - \frac{\sigma_s^2}{2})t \quad (6.23)$$

$$S_{crit} = exp[\sigma_s \sqrt{t} N^{-1}(1 - \frac{\alpha}{P(V_{1y} < D)}) + lnS_0 + (\mu_s - \frac{\sigma_s^2}{2})t] \quad (6.24)$$

 

Substitute $S_{crit}$ into the equation for Portfolio value gives

$$VaR_\alpha = g(S_{crit}) \quad (6.25)$$

$$= g(exp[\sigma_s\sqrt{t}N^{-1}(1-\frac{\alpha}{P(V_{1y}<D)}) + lnS_0 + (\mu_s - \frac{\sigma_s^2}{2})t]) \quad (6.26)$$

By substituting all the numbers into the above equation, I found that when ρ = 0, $VaR = 52.1535$

**6.1.2 ρ** = 1

For correlation coefficient ρ = 1, we can say that and let . Then by equation (6.9), we get

$$\alpha = P(V_{1y} < D, P_{1y} > VaR_\alpha) \quad (6.27)$$

$$= P(W_t < \frac{lnD - (\mu_v - \frac{\sigma_v^2}{2})t}{\sigma_v}, W_t > \frac{lnS_{crit} - lnS_0 - (\mu_s - \frac{\sigma_s^2}{2})t}{\sigma_s}) \quad (6.28)$$

$$= P(\frac{lnS_{crit} - lnS_0 - (\mu_s - \frac{\sigma_s^2}{2})t}{\sigma_s\sqrt{t}} < \frac{W_t}{\sqrt{t}} < \frac{lnD - (\mu_v - \frac{\sigma_v^2}{2})t}{\sigma_v\sqrt{t}}) \quad (6.29)$$

$$= N(\frac{lnD - (\mu_v - \frac{\sigma_v^2}{2})t}{\sigma_v\sqrt{t}}) - N(\frac{lnS_{crit} - lnS_0 - (\mu_s - \frac{\sigma_s^2}{2})t}{\sigma_s\sqrt{t}}) \quad (6.30)$$

let

$$A = N(\frac{lnD - (\mu_v - \frac{\sigma_v^2}{2})t}{\sigma_v\sqrt{t}})$$

then A is a deterministic constant. Then we can compute $S_{crit}$ and therefore $VaR_\alpha$

$$N(\frac{lnS_{crit} - lnS_0 - (\mu_s - \frac{\sigma_s^2}{2})t}{\sigma_s\sqrt{t}}) = A - \alpha \quad (6.31)$$

$$ln(S_{crit}) = \sigma_s\sqrt{t}N^{-1}(A-\alpha) + lnS_0 + (\mu_s - \frac{\sigma_s^2}{2})t \quad (6.32)$$

$$S_{crit} = exp[\sigma_s\sqrt{t}N^{-1}(A-\alpha) + lnS_0 + (\mu_s - \frac{\sigma_s^2}{2})t] \quad (6.33)$$

and finally,

$$VaR_\alpha = g(S_{crit}) \quad (6.34)$$

$$= g(exp[\sigma_s\sqrt{t}N^{-1}(A-\alpha) + lnS_0 + (\mu_s - \frac{\sigma_s^2}{2})t]) \quad (6.35)$$

By substitution, I found for ρ = 1, $VaR = 0$

**6.1.3 ρ** = −1

If the ρ = −1, we can assume .

$$\alpha = P(V_{1y} < D, P_{1y} > VaR_\alpha) \quad (6.36)$$

$$= P(\frac{W_t}{\sqrt{t}} > -\frac{lnD - (\mu_v - \frac{\sigma_v^2}{2})t}{\sigma_v\sqrt{t}} = A, \frac{W_t}{\sqrt{t}} > \frac{lnS_{crit} - lnS_0 - (\mu_s - \frac{\sigma_s^2}{2})t}{\sigma_s\sqrt{t}} = B) \quad (6.37)$$

$$= P(\frac{W_t}{\sqrt{t}} > max\{A, B\}) \quad (6.38)$$

$$= 1 - N(max\{A, B\}) \quad (6.39)$$

$$= 1 - max\{N(A), N(B)\} \quad (6.40)$$

Since α=0.01, max{N(A), N(B)}=N(B). Then

$$N(\frac{lnS_{crit} - lnS_0 - (\mu_s - \frac{\sigma_s^2}{2})t}{\sigma_s\sqrt{t}}) = 1 - \alpha \quad (6.41)$$

$$ln(S_{crit}) = \sigma_s\sqrt{t}N^{-1}(1-\alpha) + lnS_0 + (\mu_s - \frac{\sigma_s^2}{2})t \quad (6.42)$$

$$S_{crit} = exp[\sigma_s\sqrt{t}N^{-1}(1-\alpha) + lnS_0 + (\mu_s - \frac{\sigma_s^2}{2})t] \quad (6.43)$$

and finally,

$$VaR_\alpha = g(S_{crit}) \quad (6.44)$$

$$= g(exp[\sigma_s\sqrt{t}N^{-1}(1-\alpha) + lnS_0 + (\mu_s - \frac{\sigma_s^2}{2})t]) \quad (6.45)$$

Similarly, in this case I computed that for ρ = −1, *VaR* = 124.2145

### 6.2 Method 2(Monte Carlo Simulation)

Alternatively, there is another method to compute VaR. For the given portfolio, the price is given as
$Portfolio = 2BSC - BSP$

where BSC and BSP are the Black Sholes formula(Nielsen, 1992) for call and put option respectively, given as followed:

$BSC = SN(d_1) - Ke^{-r(T-t)}N(d_2)$ (6.46)

where

$$d_{1,2} = \frac{1}{\sigma\sqrt{T-t}}[\ln\frac{S}{K} + (r \pm \frac{\sigma^2}{2})(T-t)] \quad (6.47)$$

The price of a corresponding put option based on putcall parity(Nielsen, 1992) is:

$$BSP = Ke^{-r(T-t)} - S + BSC \quad (6.48)$$

$$= Ke^{-r(T-t)}N(-d_2) - SN(-d_1) \quad (6.49)$$

In this case, we only need the payoff at time t = one year. So for the Monte Carlo simulation, we only need one time interval = one year. Therefore, we can generate a large amount of standard normal deviate and substitute into the equation

$$s_{t+1} = s_t + (\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\epsilon \quad (6.50)$$

$$S_t = exp(s_t) \quad (6.51)$$

which gives the simulation for $S_T$. By substituting $S_T$ into the Black Scholes Formula, we get the price for the portfolio at one year time.

For the counterparty, we can use the same method to simulate the firm value, and the default event is triggered by the barrier D. If the counterparty does not default by one year, then there is no financial loss due to credit risk. Otherwise, if the portfolio value is positive, the loss is the positive value of the portfolio, while if the portfolio is negative, there is no loss. So the loss is given by

$$LossCR1y = \mathbf{1}_{\{V1y<D\}}(Portfolio)^+$$

Furthermore, for the correlation coefficient ρ between the Brownian motion driving the underlying stock price and the counterparty firm value, we need to use the Cholesky Decomposition(Haugh, 2004). Firstly, generate two uncorrelated standard normal variables $Z_1$ and $Z_2$. Let $X_1 = Z_1$ and $X_2$ = , then the correlation coefficient between $X_1$ and $X_2$ is .

I compute the credit VaR of the given portfolio using Monte Carlo and the procedure is as followed:

- Use standard normal (s.d) generator function to generate a pair of s.d deviates ($W_1$, $W_2$)
- Use Cholesky Decomposition to get a pair of s.d deviates ($Z_1$, $Z_2$) with correlation ρ
- By Monte Carlo simulation, substitute the above pairs of s.d deviate ($Z_1$, $Z_2$) into the equation to get a sequence of simulation of underlying stock price $S_t$ and counterparty firm value $V_t$.
- Substitute $S_1$ into the Black Scholes Formula to get the value of put (BSP) and call option (BSC) on the stock. The portfolio value is given as $Portfolio = 2BSC - BSP$.

• Find the loss due to counterparty default for each simulation. The Loss is given as

$$LossCR = \begin{cases} max(Portfolio, 0) & \text{if the counterparty default,} \\ 0 & \text{if the counterparty does not default.} \end{cases}$$

• Repeat above steps by K=10000000 times, then we get 10000000 simulations for the loss. Use the "prctile" function, we can get the VaR.

For different correlation coefficient between the Brownian motion driving the underlying stock price and the counterparty firm value, we can repeat the above procedure but use different ρ to generate the s.d ($Z_1$, $Z_2$)

Using Monte Carlo simulation for the price path, I got the result as followed:

• For ρ = -1, $VaR = 124.2442$
• For ρ = 0, $VaR = 52.1036$
• For ρ = 1, $VaR = 0$

which agrees with the result computed by method 1.

### 6.3 Numerical Analysis for the case ρ = 1 or -1

I surprisingly noticed that when ρ = 1, the VaR is 0, i.e. there will not occur any loss theoretically. The reason is that when ρ = 1

$$S_t = 100exp[(\mu_s - \frac{\sigma_s^2}{2})t + \sigma_s W_t] \quad (6.52)$$

$$V_t = exp[(\mu_v - \frac{\sigma_v^2}{2})t + \sigma_v W_t] \quad (6.53)$$

$$(6.54)$$

Then we can get

$$lnV_t = (\mu_v - \frac{\sigma_v^2}{2})t + \sigma_v W_t \quad (6.55)$$

$$W_t = \frac{lnV_t - (\mu_v - \frac{\sigma_v^2}{2})t}{\sigma_v} \quad (6.56)$$

$$S_t = 100exp[(\mu_s - \frac{\sigma_s^2}{2})t + \sigma_s \frac{lnV_t - (\mu_v - \frac{\sigma_v^2}{2})t}{\sigma_v}] \quad (6.57)$$

Therefore, we can observe that $S_t$ and $V_t$ has positive correlation. In this case, $S_t$ obtain its extreme value for maximum $V_t$. Given that the counterparty default, the maximum value of $V_t$ is $D = 0.55$. Hence, by substituting into the above equation, $max(S_t) = 78.0022$ and the portfolio value is -12.0832, indicated that the maximum value of the portfolio is negative. Thus there is no loss due to counterparty default, which verifies my result.

Similarly, for ρ = −1, $S_t$ can be represented in terms of $V_t$:

$$lnV_t = (\mu_v - \frac{\sigma_v^2}{2})t - \sigma_v W_t \quad (6.58)$$

$$W_t = \frac{(\mu_v - \frac{\sigma_v^2}{2})t - lnV_t}{\sigma_v} \quad (6.59)$$

$$S_t = 100exp[(\mu_s - \frac{\sigma_s^2}{2})t + \sigma_s \frac{(\mu_v - \frac{\sigma_v^2}{2})t - lnV_t}{\sigma_v}] \quad (6.60)$$

### 6.4 Complete Correlation Pattern Between the Brownian Motion Driven Underlying Stock Price and Counterparty Firm Value

Furthermore, I can compute the VaR using any correlation coefficient by Monte Carlo Simulation. I divided [-1,1] into 20 intervals and calculate the VaR using the starting point of each interval as the correlation coefficient. Then I plot the graph between the correlation coefficient ρ and VaR. The result can be shown by the following plot

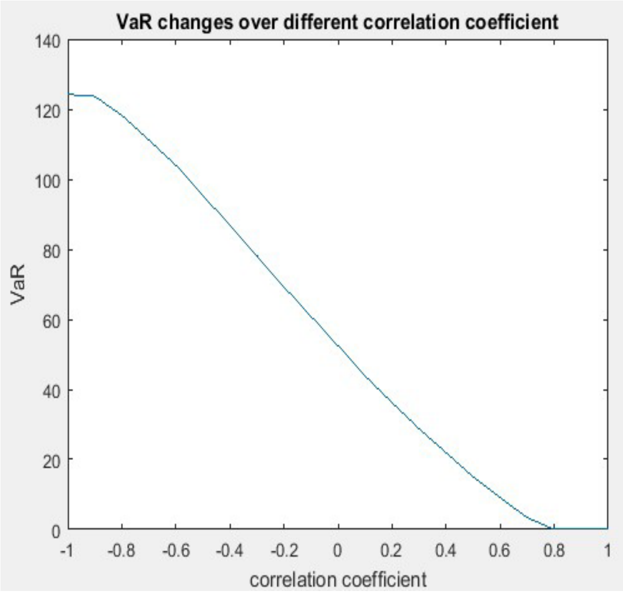


**Figure 6**
**Correlation coefficient–VaR**

Therefore, as shown above, if the correlation coefficient between the Brownian motion of underlying stock price and counterparty firm value increases, the VaR will decrease, i.e. they are negatively dependent. When finding a cooperation partner, the underlying equity should prefer the firm with positively larger correlation to itself.

## 7. ADVANCED MODELS WITH STOCHASTIC INTEREST RATES

### 7.1 Cox–Ingersoll–Ross Model

First of all, I will introduce the model interpreted by Cox, Ingersoll and Ross(CIR) in 1985(Cox and Ross, 1985). The basic assumption of this model is that under the risk-neutral measure Q, the short-term interest rate r satisfies the following equation:

 

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t \quad (7.1)$$
$$r_0 > 0 \quad (7.2)$$

where a, b, σ and $r_0$ are deterministic quantitative factors, where $a(b - r_t)$ is the drift term and is the volatility. Since for short term, when $r_t$ goes to zero, the volatility term tends to zero, avoiding the affection of randomness. In this case, it is impossible for the short term interest rate r to be negative(Haugh, 2010). According to Zeytun and Gupta, following the CIR model, the time t value of the zero-coupon bond with maturity T is:

$$P(t,T) = E[e^{-\int_t^T r_s ds}] \quad (7.3)$$
$$= e^{A(t-T)-B(t-T)r_t} \quad (7.4)$$

Where

$$A(x) = \frac{2ab}{\sigma^2}log(\frac{2\gamma e^{\frac{(\gamma+a)x}{2}}}{(\gamma+a)(e^{\gamma x}-1)+2\gamma}) \quad (7.5)$$
$$B(x) = \frac{2(e^{\gamma x}-1)}{(\gamma+a)(e^{\gamma x}-1)+2\gamma} \quad (7.6)$$
$$\gamma = \sqrt{a^2+2\sigma^2} \quad (7.7)$$

### 7.2 Kim, Ramaswamy and Sundaresan Approach

According to Kim Ramaswamy and Sundaresan(1993) (.Ramaswamy and Sundaresan, 1993), the dynamics of short-term interest rate is

$$dr_t = (a - br_t)dt + \sigma_r\sqrt{r_t}d\tilde{W}_t \quad (7.8)$$

where is a standard Brownian motion under the spot probability measure $P^*$. The model with such interest rate dynamics is referred to as the CIR term structure model (Bielecki and Rutkowski, 2002). We suppose that the risk premium for the interest rate risk is zero. Consequently, the short-term rate dynamics under the risk-neutral probability measure $P^*$ and practical probability measure P are identical. And the value process is assumed to be driven by the following SDE:

$$dV_t = V_t((r_t - \kappa)dt + \sigma_v dW_t^*)$$
(7.9)

where the Brownian motion $W^*$ and are correlated with coefficient $\rho_{V,r}$.

In this case, the bond contract prohibits the stockholder from selling the firm's assets to pay dividends. The bondholder must continuously pay a coupon at the rate of c units of currency per unit of time. The firm defaults before the maturity T if it cannot make the coupon payment, and it occurs when the stock value falls below the barrier . To be more specific, the barrier is the breakeven point such that if $V_t$= , the dividends equals the coupon payment and if $V_t$< , the dividend is not sufficient to cover the coupon payment.

As a result, we can assume that the initial value .

Given that the firm does not default before maturity, then if the price of the firm at the maturity date is lower than the notional amount K, the firm is default at the maturity (Bielecki and Rutkowski, 2002). Let $ND^c(t,T) = ND^c(t,T,r_t)$ denotes the price at time t of an non–defaultable bond with continuous payment c per unit time, and the face value K at time T. Then the payoff to the bondholder is the minimum between the firm value $V_t$ and the recovery claim $(T - t)ND^c(t,T)$(.Ramaswamy and Sundaresan, 1993).

Represent it in terms of defaultable gives: $X = K$, $A_t = c_t$, $Z = min\{V_t, \varphi(T-t)ND^c(t,T)\}$, $\tau = inf\{t\epsilon[0,T]\ V_t < v_t\}$ where is a deterministic function with (0) = 1, representing the recovery rate, and the default triggering barrier v is

$$v_t = \begin{cases} \bar{v} & \text{if t < T,} \\ K & \text{if t = T} \end{cases}$$

In financial industry, the bankruptcy cost is $V_\tau - (T - \tau) ND^c(\tau,T)$

The drawbacks of this model are the complicated expression of the triggering barrier and that practically, the volatility of a zero-coupon bond does not follow a deterministic function (Bielecki and Rutkowski, 2002).

### 7.3 Briys and de Varenne Approach

The Briys and de Varenne approach(Briys and Varenne, 1997) is a special case of the Black and Cox model with stochastic interest rate. The dynamics of the the interest rate under this model given as

$$dr_t = a(t)(b(t) - r_t)dt + \sigma(t)d\tilde{W}_t$$
(7.10)

where a(t), b(t), σ($t$) : $[0,T] \rightarrow$ R are deterministic functions. Consequently, the price of a non-defaultable zero-coupon bond is

$$dND(t,T) = ND(t,T)(r_t dt + b(t,T)d\tilde{W}_t)$$
(7.11)

for some deterministic function $b(\cdot,T)$ $[0,T] \rightarrow$ R. The value of the firm is defined by

$$\frac{dV_t}{V_t} = r_t dt + \sigma_v(\rho d\tilde{W}_t + \sqrt{1-\rho^2}d\hat{W}_t)$$
(7.12)

where $\sigma_v > 0$ is a deterministic quantitative factor, and are mutually independent Brownian motions, and $\rho = \rho_{V,r}$ is the correlation coefficient between the interest rate and the firm's value.

The default barrier is

$$v_t = \begin{cases} kND(t,T) & \text{if t} < \text{T}, \\ K & \text{if t} = \text{T}. \end{cases}$$

where K is the notional amount at time T and k is a constant with $0 < k \leq K$.

And the default time is

$$\tau = inf\{t\epsilon[0,T]\ V_t < v_t\}$$

(7.13)

The payoff to the bondholder given default is defined as

$$Payoff = \begin{cases} \beta_2 V_\tau & \text{if } \tau < \text{T} \\ \beta_1 V_T & \text{if } \tau = T \\ K & \text{if } \tau > \text{T} \end{cases}$$

where $\beta_1$ and $\beta_2$ are the recovery rate at default for $\tau < T$ and $\tau = T$ respectively. In terms of defaultable claim, the above model can be represented as (Bielecki and Rutkowski, 2002)

$$X = K,\ A = 0,\ Z = \beta_2 V_\tau,\ \tilde{X} = \beta_1 V_T,\ \tau = inf\{t\epsilon[0,T]\ V_t < v_t\}$$

## REFERENCES

Armstrong, J. (2015). *Numerical and Computational Methods in Finance*. Computational and Numerical Methods for Mathematical Finance MATLAB programming and numerical methods for KCL MSc students. King's College London Link.

Basurto, M. S., & Singh, M. (2008). *Counterparty Risk in the Over-the-Counter Derivatives Market, November. IMF Working Papers*, 1–19, Available at SSRN: http://ssrn.com/abstract¼1316726.

Bielecki, T. R., & Jeanblanc, M. (2004). *Indifference Pricing of Defaultable Claims. In Indifference pricing, Theory and Applications, Financial Engineering*. Princeton University Press.

Bielecki, T. R., & Rutkowski, M. (2002). *Credit risk: Modeling, valuation and hedging*. Springer-Verlag, Berlin Heidelberg New York.

Bielecki, T. R., & Rutkowski, M. (2004). *Credit Risk: Modeling, Valuation and Hedging.* Berlin: Springer-Verlag..

Breccia, A. (2012). Default Risk in Mertons Model. http://www.bbk.ac.uk/ems/forstudents/mscfinEng/pricingemms014p/ab8.pdf

*Briys*, *E*. and de *Varenne*, *F*. (*1997*). *Valuing risky fixed rate debt*: An *extension*. *Journal of Financial and Quantitative Analysis, 32*, 239-248.

Cox, J., Ingersoll, J., Jr., & Ross, S. (1985). A theory of the term structure of interest rates. *Econometrica, 53*(2), Mach, 385-407.

Crouhy, M., Galai, D., & Mark, R. (2000). A Comparative Analysis of Current Credit Risk Models. *Journal of Banking & Finance, 24*, 59-117.

Gueant, O. (2012). *Computing the value at risk of a portfolio: Academic literature and practionners'response*. EMMA, Working Paper.

Haugh, M. (2004). The Monte Carlo Framework, Examples from Finance and Generating Correlated Random Variables. In *Monte Carlo Simulation Course Notes* (IEOR, 2004).

Haugh, M. (2010). *Continuous-Time Short Rate Models*. Financial Engineering: Continuous-Time Models

Haugh, M. (2010). *Risk Measures*, *Risk Aggregation and Capital* Allocation. IEOR E4602. Quantitative *Risk* Management.

Hazewinkel, & Michiel. (2001). Central Limit Theorem. In *The Concise Encyclopedia of Statistics* (pp 66-68).

Hull, J. (1993). *Options, Forward Contracts, Swaps and Other Derivatives Securities.* Prentice Hall.

Jarrow, R. A., & van Deventer, D., & Wang, X. (2003). A robust test of Merton's structural model for credit risk. *J. Risk, 6,* 39-58.

Johnson, H., & Stulz, R. (1987). The pricing of options with default risk. *The Journal of Finance, 42(2), 267–280*

Jorion, P. (2007). *Value at Risk: The New Benchmark for Managing Financial Risk*. McGraw Hill.

Kim, I.J., Ramaswamy, K.V., & Sundaresan, S. (1993). Does Default Risk in Coupons Affect the Valuation of Corporate Bonds?: A Contingent Claims Model. *Financial Management, 22,* 117-131. DOI:10.2307/3665932

Kondapaneni, R. (2005). *A study of the Delta Normal Method of Measuring VaR*. Worcester Polytechnic Institute. INTERNET. https://digitalcommons.wpi.edu/etdtheses/793/ [Accessed 10 November 2019].

Kyng, T. J., & Konstandatos, O. (2014). Multivariate Monte-Carlo Simulation and Economic Valuation of Complex Financial Contracts: An Excel-Based Implementation. *Spreadsheets in Education, 7* (2). Retrieved from http://epublications.bond.edu.au/ejsie/vol7/iss2/5/

Li, D. X. (1999). On Default Correlation: A Copula Function Approach. *The Journal of Fixed Income, 9*(4), 43-54.

Mcneil, A. J., Frey, R., & Embrechts, P. (2005*). Quantitative Risk Management: Concepts, Techniques, and Tools*. Princeton University Press.

Nielsen, L. T. (1992). Understanding N(d1) and N(d2): Risk–Adjusted Probabilities in Black–Scholes Models.

Nielsen, L.T. (1992). *Understanding N(d1) and N(d2): Risk-Adjusted Probabilities in the Black-Scholes Model*. INSEAD. Available online: https://financetrainingcourse.com/education/wp-content/uploads/2011/03/Understanding.pdf (accessed on 7 May 2021).

Rachev, S. (2009). *Credit Risk: Intensity Based Model. Institute for Statistics and Mathematical Economics.* University of Karlsruhe and Karlsruhe Institute of Technology (KIT).

 

Schoutens, W., & Cariboni J. (2009). *Levy Processes in Credit Risk.* Wiley.

Segoviano, M. A., & Singh, M. (2008). Counterparty.

Weisstein, E. W. (2004). *Taylor Series.* Mathworld. http://mathworld.wolfram.com/TaylorSeries.html

Zeytun, S., & Gupta, A. (2001). *A Comparative Study of the Vasicek and the CIR Model of the Short Rate. Fraunhofer-Institut für Techno- und Wirtschaftsmathematik*, Available at: www.itwm.fraunhofer.de/zentral/download/berichte/bericht124.pdf.