CSCanada
Management Science and Engineering
Vol. 14, No. 2, 2020, pp. 5-8
DOI:10.3968/12136

ISSN 1913-0341 [Print]
ISSN 1913-035X [Online]
www.cscanada.net
www.cscanada.org

Application of Ordinary Differential Equation in Practice

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Supported by the Jiangxi Province Educational Reform in 2019: Exploration and practice of the Integration of Five Micros comprehensive teaching model of Advanced Mathematics. Serial number: JXJG-19-2-20.

Received 15 August 2020; accepted 3 October 2020
Published online 26 December 2020

Abstracts

Among relationships between variables reflecting movement process in natural, there are many mathematical models that satisfy ordinary differential equation (ODE). Generally, as an important tool for solving many problems in practice, ODE is used to understand property of an unknown function. This paper describes four practical cases that apply ODE, i.e. identification of death time in criminal investigation; Malthusian population model; archaeology of Mawangdui Han Dynasty Tomb in Changsha (Hunan Province); and authentication of oil painting, etc.

Key words: Ordinary differential equation (ODE); Criminal investigation; Authentication for oil painting; Population model; application

Gui, G. X., & Yang, S. L. (2020). Application of Ordinary Differential Equation in Practice. *Management Science and Engineering*, *14*(2), 5-8. Available from: URL: http://www.cscanada.net/index.php/mse/article/view/12136 DOI: http://dx.doi.org/10.3968/12136

INTRODUCTION

In the world, many natural phenomena with certain laws can be explained by ODE. Many theories and rules of each field can be described by ODE, for example, the law of universal gravitation, law of conservation of mechanical energy, and the law of conservation of energy in physics, and genetic variation in biology, etc. The knowledge can be understood, learnt and analyzed through corresponding studies on ODE. Therefore, ODE related knowledge can be used not only in natural science, but also in practical life.

Definition 1 (Wang, et, al., 2006): Any equation that can indicate the relationship among an unknown function, derivative and independent variable of an unknown function is called as differential equation. If an unknown function is unary, it is called as ODE. If an unknown function is multivariate, it is called as partial differential equation (PDE).

Definition 2 (Wang, et, al. 2006): The order of highest derivative of an unknown function occurred in a differential equation is called as order of the differential equation.

Definition 3 (Ding & Li, 2000): If a differential equation can be identical when a certain function is substituted, then this function is called as solution of the differential equation. If the number of any independent constants contained in solution of the differential equation is same with orders, then this solution is called as general solution of the differential equation.

1. APPLICATION OF ORDINARY DIFFERENTIAL EQUATION (ODE) IN PRACTICE

1.1 Identification of Death Time in Criminal Investigation

ODE related knowledge can also be seen in Newton's Law of Cooling which is used to solve some practical problems. Therefore, ordinary differential knowledge is used indirectly. Newton's Law of Cooling indicates that the cooling rate of an object in the air is in direct proportion to the difference between object temperature and air temperature.

If an object is stored in a room that is relatively larger compared to the object, then the room temperature is considered being constant, i.e. normal temperature. In other words, change of the room temperature due to this object can be neglected. If a murder case occurred, then the body temperature began to decline from 37°C before death, which was consistent with the Newton's Law of Cooling. If the ambient temperature was constant, being 20℃, then the body temperature would change into 35℃ two hours later. If a body with temperature 30℃ was found at 4:00 PM, then death time could be inferred based on above information. According to equation $\begin{cases} \frac{dH}{dt} = -k(H-20), \\ H(0) = 37, \end{cases}$, $H - 20 = Ce^{-kt}$ can be obtained through separation of variables. According to initial value condition $H(0)=37$, C=17 can be obtained; as a result, solution of the initial value problem is $H = 20+17e^{-kt}$.

Based on above conditions, body temperature changed into 35°C two hours later, so that k could be calculated based on $35 = 20 + 17e^{-2k}$, being $k \approx 0.063$. Hence, the temperature reduction time t could be calculated, being $t \approx 8.4$ by substituting temperature $H = 30$ into the temperature function $H = 20 + 17e^{-0.063t}$. As a result, it can be inferred that the murder case occurred 8.4 hours before the body was founded at 4:00 PM, i.e., 7:36 AM.

1.2 Malthusian Population Model

Although it is discrete for increase and decrease of population, the increment and decrement among few individuals are very small compared to the total population. Therefore, change of large-scale population size with time is approximately continuous or even slight. In order to study this phenomenon conveniently, ODE is used here.

Given that $y(t)$ is population size of a certain region at time t and $k(t)$ is the difference between birth rate and mortality rate. If people here are isolated, that is to say, immigration and emigration cannot be seen, then change rate of population $y'(t) = ky(t)$ can be obtained. Mostly, it is assumed that k is a constant. That is to say, k will not change with time. As a result, equation $\frac{dy}{dt} = ky$ can be obtained, which is called as Malthus Theory in demography. $y = ce^{kt}$ can be obtained by this equation, where c is an arbitrary constant. If population size of a certain region is y_0 at time t_0, then $y_0 = ce^{kt_0}$ can be obtained. Besides, $c = y_0e^{-kt_0}$ is substituted into $y_0 = ce^{kt_0}$ to get $y(t) = y_0e^{k(t-t_0)}$. This function indicates that increase of population size under given conditions complies with the law of index.

The population size from 1860 to 1870 in USA can be estimated by above equation, which is approximate to the value obtained through statistics of the year. However, it is not feasible to compare with value of the year when population size from 1870 to 1990 in USA is estimated. In fact, it is impossible to occur from the view of biology if $t \to +\infty$. Because population increment can be influenced by objective conditions such as population mobility, birth, death, illness, and old age. Due to these conditions, the population cannot grow as per above rules. As a result, it is proper to assume that relative population growth rate is a constant if the time gap is short. But the relative growth rate is not a constant if time gap is extended. In consideration of various factors, biomathematician Verhulst from Netherland obtained equation set $\begin{cases} \frac{dp}{dt} = (a-bp)p \\ p(t_0) = p_0 \end{cases}$ in 1837, where a,b are constant and called as life constant. If $p \neq 0$ or $p \neq \frac{a}{b}$, integral $\int_{p_0}^{p} \frac{du}{u(a-bu)} = \int_0^t d$ can be obtained after separation of variables because of $\frac{1}{u(a-bu)} = \frac{1}{a}(\frac{1}{u} + \frac{1}{a-bu})$, hence,

$$\frac{1}{a}\ln\frac{u}{a-bu}\Big|_{p_0}^{p} = t - t_0$$, i.e. $$\ln\frac{p(a-bp_0)}{p_0(a-bp)} = a(t-t_0)$$,

and then $\frac{p}{a-bp} = \frac{p_0e^{a(t-t_0)}}{a-bp_0}$. According to historical data, this equation was used in USA and France to predict demographic change and results were consistent with practical conditions.

1.3 Archaeology of Mawangdui Han Dynasty Tomb in Changsha (Hunan Province)

According to the theory of atomic physics, disintegration rate of radioisotope C-14 (denoted by ^{14}C) is in direct proportion to ^{14}C content at the time t. Alive creatures continuously take ^{14}C from their bodies (with same ^{14}C content in air) and immediately stop after their death, and then disintegration begins for ^{14}C in dead body. ^{14}C content change in dead body with time t is studied if ^{14}C content at the moment of death is x_0, and the year of No.1 Tomb from Mawangdui Han Dynasty Tomb is inferred by this rule.

If ^{14}C content in body of a creature is x at the time of death and $x(t)$ at time t, then $\begin{cases} \frac{dx}{dt} = -kx \\ x|_{t=0} = x_0 \end{cases}$ can be obtained, where $k > 0$ is a proportional constant and negative sign indicates that ^{14}C content decreases continuously. If the half-time period of ^{14}C is T, i.e. $x(T) = \frac{x_0}{2}$, then $t = \frac{T}{\ln 2} \ln \frac{x'(0)}{x'(t)}$ can be obtained. Given that half-time period of ^{14}C is known, being $T = 5568$, measured average atomic disintegration rate is $x'(t) = 29.78$ times/min for ^{14}C in unearthed charcoal sample and $x'(0) = 38.37$ times/min for ^{14}C in charcoal from new woods, then $t = \frac{5568}{\ln 2} \ln \frac{38.37}{29.78} \approx 203$ (years). Therefore, it can be inferred that No. 1 Tomb was a Han dynasty tomb 2036 years before.

1.4 Authentication of Oil Painting

How to authenticate an artwork? What is the relationship with mathematics? In fact, this problem can be solved through linear first-order differential equation. How to transform an irrelevant problem into a mathematical problem? The modelling is implemented below to solve this problem.

After World War II, the Security Department of Field Army of Netherland found a Vermeer's oil painting in the house of a Nazi officer Gorlin and investigated that it was sold to Gorlin by modern painter Malines. Hence, Malines was arrested, but he stated in court that the painting was fake. The crime of treason or forgery could not be determined by the judge. Finally, it evolved into a case of authenticating the artwork. An authentication group that was constituted by chemist, physicist and artist investigated and analyzed the painting from the view of production time of some pigments and artistic style, and finally determined it being counterfeit. But it was not convincing. In 1967, a scientific research team from Carnegie Mellon University demonstrated through radiation cycle of element Pb in the painting that it had a history of only decades of years instead of over three hundred years.

What is the scientific theory of authenticating above artwork? In fact, with respect to the key of determining history of an oil painting, the well-known physicist Rutherford who found radioactivity at the beginning of the last century stated that disintegration rate of radioactive element was in direct proportion to content of atoms without disintegration. If number of atoms at time t is $N(t)$, then $N(t)$ is a discrete function, but number of atoms is extremely large and its disintegration is very slowly. Therefore, $N(t)$ can be considered as a continuous differentiable function to get $\frac{dN}{dt} = -\lambda N$, where λ is a disintegration constant. For this practical problem, there is an initial condition $N(t)|_{t=t_0} = N_0$. As a result, $N(t) = N_0 e^{-\lambda(t-t_0)}$ can be obtained by separation of variables, and then age of an object can be calculated by equation $t - t_0 = \frac{1}{\lambda} \ln \frac{N_0}{N}$.

In respect of radioactivity, what are radioactive elements in an oil painting? What is the relationship among these radioactive substances? The main ingredient of an oil painting is white lead that consists of Pb-210 and Ra-226. Ra-226 will generate new Pb-210 during disintegration. As a result, Pb-210 content in an oil painting depends on its own disintegration and supplement of Ra. Hence, authenticity of an oil painting can be determined through measuring Pb-210 content and determining year of the painting. This problem can be transformed into building a mathematical model for Pb-210 content.

Model calculation: The method of variation of constant is used to obtain $N(t) = \frac{R}{\lambda}[1 - e^{-\lambda(t-t_0)}] + N_0 e^{-\lambda(t-t_0)}$ and calculation equation for age of oil painting $t - t_0 = \frac{1}{\lambda} \ln \frac{\lambda N_0 - R}{\lambda N - R}$, where λ is known and $\lambda N(t), R$ can be calculated. For example, Carnegie Mellon University measured several oil paintings and obtained data as below:

Name of oil painting	$\lambda N(t)$ (Piece/min)	R (Piece/min)
1.Jesus And His Disciples	8.5	0.8
2.Washing Feet	12.6	0.26
3.The Woman Reading Music Score	10.3	0.3
4.The Lacemaker	1.5	1.4
5.The Smile Woman	5.2	6.0

If initial value N_0 is known, the above age calculation can be calculated. In fact, it is impossible to get N_0. But it does not mean that this model is invalid. In other words, if Pb disintegration limit is 30,000, i.e. , the above equation can be transformed into $\lambda N_0 = \lambda N(t)e^{\lambda(t-t_0)} - R[e^{\lambda(t-t_0)} - 1]$.

Model results: If the artwork "Jesus And His Disciples" is authentic, then it has a history of 300 years, i.e. $t - t_0 \approx 300$. Based on Pb-210 disintegration constant $\lambda = \frac{\ln 2}{22}$, $\lambda N(t) = 8.5$ piece/min and $R = 0.8$ piece/min, the value

$\lambda N_0 \approx \lambda N(t)e^{300\lambda} - R[e^{300\lambda} - 1] \approx 98053$ can be calculated, which is greatly different from given value 30000. In other words, "Jesus And His Disciples" is counterfeit definitely, and then authentication results of other paintings in above Table can be obtained as below:

Name of oil painting	Measured λN_0 (piece/min)	Results
1.Jesus And His Disciples	95053	Counterfeit
2.Washing Feet	157138	Counterfeit
3.The Woman Reading Music Score	127340	Counterfeit
4.The Lacemaker	1274	Counterfeit with history no more than decades of years
5.The Smile Woman	-10181	

It can be inferred by measured λN_0 that the first three paintings are definitely counterfeit. What about the 4th and 5th painting? It can be only determined that they are not counterfeit over the past few years instead of their authenticity.

CONCLUSIONS

ODE is useful for authenticating counterfeit. Although authenticity of the last two paintings cannot be determined, analysis for them is useful to a certain extent. The differential equation is widely used in practice, for example, hydrochloric content in polluted lake can be determined by differential equation, which shall be further studied and analyzed in the future.

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