Therefore,

$$\xi(\{v_n\}) \le \left(4M\|\mu\|_{L(J;\mathbb{R}^+)} + M\sum_{k=1}^{m} l_k\right)\xi(\{v_n\}).$$

Together with (4), this allow us to conclude that

$$\xi(\{v_n\}) = 0. \qquad (8)$$

Let $t_1', t_2' \in J \setminus \{t_k : k = 1,...,m\}$ with $t_1' < t_2'$ and $\theta > 0$ such that $\{t_k : k = 1,...,m\} \cap \{[t_i' - \theta, t_i' + \theta] : i = 1,2\} = \varnothing$. It is easy to see that

$$\| T(t_2')u_0 - T(t_1')u_0 \| \to 0 \quad \text{as } t_2' - t_1' \to 0.$$

Let $\tau > 0$ be small enough, we have that

$$\begin{aligned}
&\left\|\int_0^{t_2'} T(t_2'-s)f_n(s)\mathrm{d}s - \int_0^{t_1'} T(t_1'-s)f_n(s)\mathrm{d}s\right\| \\
&\le \sup_{s\in[0,t_1'-\tau]} \| T(t_2'-s) - T(t_1'-s)\|_{L(X)} \int_0^{t_1'-\tau} \| f_n(s)\| \,\mathrm{d}s \\
&\quad + 2M\int_{t_1'-\tau}^{t_1'} \| f_n(s)\| \,\mathrm{d}s + M\int_{t_1'}^{t_2'} \| f_n(s)\| \,\mathrm{d}s \\
&\to 0 \quad \text{as } t_2' - t_1' \to 0, \tau \to 0,
\end{aligned}$$

and it follows from (H_4) that

$$\begin{aligned}
&\left\|\sum_{0<t_k<t_2'} T(t_2'-t_k)I_k(v_n(t_k)) - \sum_{0<t_k<t_1'} T(t_1'-t_k)I_k(v_n(t_k))\right\| \\
&\le \sum_{0<t_k<t_1'} c_k \| T(t_2'-t_k) - T(t_1'-t_k)\|_{L(X)} + M\sum_{t_1'\le t_k<t_2'} c_k \\
&\to 0 \quad \text{as } t_2' - t_1' \to 0
\end{aligned}$$

Here, we have used the continuity of $T(t)$ for $t > 0$ in the uniform operator topology. Therefore, one gets for each k=0,...,m that

$$\| \mathcal{P}_k(v_n)(t_2') - \mathcal{P}_k(v_n)(t_1') \| = \| v_n(t_2') - v_n(t_1') \| \to 0 \quad \text{as } t_2' - t_1' \to 0$$

uniformly for $\| \mathrm{P}_k(v_n) \in \mathrm{D}$.

Now, we consider the case when $t=t_k, k=1,...,m$. Fix $\delta_2>0$ such that $\{t_i : i \ne k\} \cap [t_k - \delta_2, t_k + \delta_2] = \varnothing$ one has

$$\begin{aligned}
&\| \mathcal{P}_{k-1}(v_n)(t_k) - \mathcal{P}_{k-1}(v_n)(t_k - \gamma) \| \\
&= \| v_n(t_k) - v_n(t_k - \gamma) \| \\
&\le \| (T(t_k) - T(t_k - \gamma))u_0 \| \\
&\quad + \sup_{s\in[0,t_k-\gamma]} \| T(t_k - s) - T(t_k - \gamma - s) \|_{\mathcal{L}(X)} \int_0^{t_k-\gamma} \| f_n(s) \| \,\mathrm{d}s \\
&\quad + M\int_{t_k-\gamma}^{t_k} \| f_n(s) \| \,\mathrm{d}s + \sum_{i=1}^{k-1} c_i \| T(t_k - t_i) - T(t_k - \gamma - t_i) \|_{\mathcal{L}(X)} \\
&\to 0 \quad \text{as } \gamma \to 0.
\end{aligned}$$

Also,

$$\begin{aligned}
&\| \mathcal{P}_k(v_n)(t_k + \gamma) - \mathcal{P}_k(v_n)(t_k) \| \\
&= \| v_n(t_k + \gamma) - v_n(t_k^+) \| \\
&= \| v_n(t_k + \gamma) - v_n(t_k) - I_k(v_n(t_k)) \| \\
&\le \| (T(t_k + \gamma) - T(t_k))u_0 \| \\
&\quad + \sup_{s\in[0,t_k]} \| T(t_k + \gamma - s) - T(t_k - s) \|_{\mathcal{L}(X)} \int_0^{t_k} \| f_n(s) \| \,\mathrm{d}s \\
&\quad + M\int_{t_k}^{t_k+\gamma} \| f_n(s) \| \,\mathrm{d}s + \sum_{i=1}^{k} c_i \| T(t_k + \gamma - t_i) - T(t_k - t_i) \|_{\mathcal{L}(X)} \\
&\to 0 \quad \text{as } \gamma \to 0.
\end{aligned}$$

$$\mathrm{mod}_{\mathrm{C}}(\{\mathcal{P}_k(v_n)\}) = 0, \quad k = 0,...,m. \qquad (9)$$