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Modifying Weak Solutions of a Triangular Fuzzy Linear System to Strong Ones

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Abstract: This paper is concerned with the structure of solution space of triangular fuzzy linear systems. The existence of non-triangular fuzzy number solutions for triangular fuzzy linear systems is proved. According to the structure of solution space, an approach of modifying a weak fuzzy number solution of the triangular fuzzy linear system to a strong solution is illustrated.

Key Words: Fuzzy numbers; Solution space; Fuzzy linear system

1. INTRODUCTION

Systems of simultaneous linear equations play center role in various areas such as mathematics, physics, statistics, and so on. But in many problems the parameters in the systems are not known exactly and they are represented by fuzzy numbers. It is important to study the structure of solution space of fuzzy linear systems[1] from the existence of solution to numerical methods. Since triangular fuzzy numbers are widely used in engineering, we primarily concentrate on triangular fuzzy linear systems. An embedding approach for solving fuzzy linear system $Ax = y$ was first proposed in [2, 3], where A is a crisp $n \times n$ matrix, x and y are fuzzy vectors. The approach transfers a fuzzy linear system into a crisp functional linear system. By solving the crisp function linear system, the solution of original fuzzy linear system can be obtained either in strong or weak senses. The embedding approach is employed to investigate the structure of solution to triangular fuzzy linear systems in this paper. Then a method of modifying a weak solution of triangular fuzzy linear system to a strong solution is given.

The structure of this paper is organized as follows. In Section 2, some definitions and results on fuzzy linear system are introduced. The existence of weak/strong solutions to triangular fuzzy linear systems is explored in Section 3, followed by the structure of solution to functional linear system $SX = Y$ in Section 4. An approach of modifying a weak solution of triangular fuzzy linear systems to a strong solution is illustrated in Section 5, and the concluding remarks are related in Section 6.

2. PRELIMINARIES

In this section we recall the basic notations of fuzzy number arithmetic and fuzzy linear system.

Let $\mathfrak{E}$ be the set of fuzzy numbers defined on $\mathbb{R}$. A fuzzy number is uniquely characterized by its cuts. The $r-$cuts of fuzzy number u are denoted by $[u]^r$. Let $u \in \mathfrak{E}$, write $[u]^r = [\underline{u}(r), \overline{u}(r)], r \in [0, 1]$. Then

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$\underline{u}(r), \overline{u}(r)$ can be regarded as functions on $[0, 1]$, which satisfy the following requirements[4]:

(i) $\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0, 1]$.

(ii) $\overline{u}(r)$ is a bounded left continuous non-increasing function over $[0, 1]$.

(iii) $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

(iv) $\underline{u}(r), \overline{u}(r)$ are right continuous at $r = 0$.

Conversely, for any functions $a(r)$ and $b(r)$ defined on $[0, 1]$ which satisfy (i)-(iv) on the above, there exists a unique $u \in \mathfrak{E}$ such that $[u]^r = [a(r), b(r)]$ for all $r \in [0, 1]$.

A fuzzy number $u \in \mathfrak{E}$ is called a triangular fuzzy number if its sendograph is a triangular, so it is uniquely characterized in parametric form by

$$\underline{u}(r) = u_c - (1 - r)(u_c - u_l), \overline{u}(r) = u_c + (1 - r)(u_r - u_c), r \in [0, 1], \tag{1}$$

where u_c, u_l and u_r are the mean value, left and right spreads of u, respectively.

The addition and scalar multiplication of fuzzy numbers previously defined can be described as follows, for arbitrary $u = [\underline{u}(r), \overline{u}(r)], v = [\underline{v}(r), \overline{v}(r)]$ and real number λ,

(a) $u + v = [\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)]$;

(b) $\lambda u = \begin{cases} [\lambda \underline{u}(r), \lambda \overline{u}(r)], & \lambda \geq 0, \\ \left[\lambda \overline{u}(r), \lambda \underline{u}(r)\right], & \lambda < 0. \end{cases}$

Definition 2.1 *The $m \times n$ linear system*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = y_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = y_2, \\ \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = y_m, \end{cases} \tag{2}$$

is called a fuzzy linear system (FLS in short)[3], where the coefficients matrix $A = (a_{ij})$ is a crisp $m \times n$ matrix and y_i is a fuzzy number for $i = 1, ..., n$. Moreover, if all the y_i at the right-hand side are triangular fuzzy numbers, (2) is a triangular fuzzy linear system (or TFLS).

Let $x_j = [\underline{x}_j(r), \overline{x}_j(r)], j = 1, ..., n$ and $y_i = [\underline{y}_i(r), \overline{y}_i(r)], i = 1, ..., m$ be fuzzy numbers. Then fuzzy linear system (2) can be rewritten in the form of following functional linear system:

$$\begin{cases} \sum_{j=1}^{n} \underline{a_{ij}x_j} & = \underline{y}_i, i = 1, ..., m, \\ \sum_{j=1}^{n} \overline{a_{ij}x_j} & = \overline{y}_i, i = 1, ..., m. \end{cases} \tag{3}$$

The functional linear system (3) can be represented in $2m \times 2n$ vector-matrix form as follows:

$$SX = Y, \tag{4}$$

or in partitioned form,

$$\begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix} \begin{pmatrix} \underline{X} \\ -\overline{X} \end{pmatrix} = \begin{pmatrix} \underline{Y} \\ -\overline{Y} \end{pmatrix}, \tag{5}$$

where $X = (\underline{X}, -\overline{X})^T$ and $Y = (\underline{Y}, -\overline{Y})^T$ and s_{ij} are defined as

$$a_{ij} \geq 0 \quad \Rightarrow \quad s_{ij} = s_{i+n,j+n} = a_{ij},$$
$$a_{ij} < 0 \quad \Rightarrow \quad s_{i+n,j} = s_{i,j+n} = -a_{ij},$$

and any s_{ij} which is not determined is zero such that $A = S_1 - S_2$. In Eq. (5), $\underline{X}$ and $\overline{X}$ denote $(\underline{x}_1, ..., \underline{x}_n)^T$ and $(\overline{x}_1, ..., \overline{x}_n)^T$, respectively; similarly, $\underline{Y}$ and $\overline{Y}$ are defined.

Definition 2.2 *Let* $\hat{X} = \{(\underline{x}_i(r), \overline{x}_i(r)), 1 \leq i \leq n, r \in [0, 1]\}$ *be the solution of Eq. (5), the fuzzy number vector* $\hat{Y} = \{(\underline{u}_i(r), \overline{u}_i(r)), 1 \leq i \leq n, r \in [0, 1]\}$ *defined by*[1]

$$\begin{aligned} \underline{u}_i(r) &= \min\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1), \overline{x}_i(1)\}, \\ \overline{u}_i(r) &= \min\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1), \overline{x}_i(1)\}, \end{aligned}$$

is called the fuzzy solution of Eq. (2). If $(\underline{x}_i(r), \overline{x}_i(r))$ *is a fuzzy number for* $i = 1, ..., n$ *then* $\hat{Y}$ *is called a strong fuzzy solution, otherwise, a weak solution.*

Weather a fuzzy linear system has strong solution depends on not only the coefficients matrix A but also the right-hand fuzzy vector[2]. We could get a fuzzy solution of Eq. (2) by using iterative or eliminating methods to solve Eq. (5). However, the derived solution may be a weak one even though the fuzzy linear system is consistent. It is well-known that the consistent under-determined crisp (not fuzzy) linear system has many solutions and this solutions constitute a linear manifold in $\mathbb{R}^n$. We next consider the existence of weak/strong fuzzy solutions to TFLS and the structure of solution space to functional linear system $SX = Y$.

3. EXISTENCE OF WEAK/STRONG SOLUTIONS TO TFLS

A triangular fuzzy linear system has either triangular fuzzy number solution or nontriangular fuzzy number solution as the next examples show.

Example 1. The 1×2 TFLS

$$x_1 + x_2 = (2r, 4 - 2r), \tag{6}$$

has a nontriangular fuzzy number solution $x^* = (x_1^*, x_2^*)$ and a triangular fuzzy number solution $\hat{x} = (\hat{x}_1, \hat{x}_2)$, where $x_1^* = (r^2 - 1, 1 - r^2)$, $x_2^* = (1 + 2r - r^2, 3 - 2r + r^2)$, $\hat{x}_1 = \hat{x}_2 = (r, 2 - r)$.

However, the next example shows that a TFLS has possibly no triangular fuzzy vector solution at all.

Example 2. The 2×2 TFLS

$$\begin{cases} x_1 - x_2 = (r, 4 - 2r), \\ -x_1 + x_2 = (r, 2 - r), \end{cases} \tag{7}$$

has no triangular fuzzy number solution. Actually this TFLS is not consistent (see Theorem 4.6, Section 4).

A question naturally arises: has the triangular TFLS (7) non-triangular fuzzy numbers solutions? The following theorem gives a negative answer.

Theorem 3.1 *If a TFLS has nontriangular fuzzy number solution, then it has triangular fuzzy number solution as well.*

The proof will be given after Theorem 4.1 (see Section 4).

Theorem 3.2 *The necessary condition that TFLS* $Ax = y$ *has nontriangular fuzzy number solution is* $\mathtt{rank} S < 2n$, *where* S *is defined by Eq. (4).*

Proof. Since S is a $2m \times 2n$ matrix, $\mathtt{rank}S \leq \min\{2n, 2m\}$. Now assume that $\mathtt{rank}S = 2n$. Deleting the superfluous rows from $SX = Y$ leads to $S'X = Y'$ which has the same solution to $SX = Y$, where X, Y are defined by Eq. (5). It follows that $\mathtt{rank}S' = 2m = 2n$. Therefore, if the functional linear system $S'X = Y'$ has solution, then the solution, denoted by $(S')^{-1}Y'$, is unique. Since all the entries in y are triangular fuzzy numbers, Y consists of linear functions, so does Y'. If fuzzy vector x satisfies TFLS $Ax = y$, then X corresponding to x by Eq. (5) is a solution of $SX = Y$ and $S'X = Y'$. It is concluded that if TFLS has fuzzy number solution, all the entries of the solution must be triangular fuzzy numbers. □

Theorem 3.4 implies that only under-determined TFLS possibly has non-triangular fuzzy number solution and over-determined (i.e. $\mathtt{rank}S = 2n > 2m$) or proper-determined (i.e. $\mathtt{rank}S = 2n = 2m$) TFLS has not.

4. STRUCTURE OF SOLUTION SPACE OF FUNCTIONAL LINEAR SYSTEM $SX = Y$

Denote $\overline{C}^n[0,1]$ the family of all the vector-valued functions which satisfy: (a) they are bounded left-continuous on $(0,1]$, (b) they have right limit on $[0,1)$ and (c) they are right continuous at $r = 0$. The supremun norm for $\overline{C}^n[0,1]$ is defined by $\|u\| = \sup_{r\in[0,1]} |u|$, where $u \in \overline{C}^n[0,1]$ and $|u|$ is Euclidian norm of u, i.e. $|u| = (u^T u)^{1/2}$. It is obvious that $(\overline{C}^n[0,1], \|\cdot\|)$ is a Banach space, so is product space $\overline{C}^n[0,1]\times\overline{C}^n[0,1]$ with norm $\|(\cdot,\cdot)\| = \max\{\|\cdot\|, \|\cdot\|\}$.

The n-dimension fuzzy vector space[5] can be isomorphically embedded[4] in $\overline{C}^n[0,1] \times \overline{C}^n[0,1]$. By fuzzy vector we mean all entries of the vector are fuzzy numbers. If fuzzy vector x satisfies FLS $Ax = y$, the function vector Y by Eq. (5) is a solution of $SX = Y$. The converse, however, does not hold since if $X = (\underline{X}, \overline{X})^T$ corresponds to a fuzzy vector, then $\underline{X} \leq \overline{X}$ must be fulfilled.

Theorem 4.1 *Suppose that* $\{\alpha_i\}_{i=1}^s$ *is a group of nonlinear functions such that* $\{\alpha_i\}_{i=1}^s \cup \{1, r\}$ *is linearly independent. There are solutions to* $SX = Y$ *in space* $(\mathrm{span}\{1, r, \alpha_1, \cdots, \alpha_s\})^{2n}$ *if and only if* $\exists Z_1 \in (\mathrm{span}\{1, r\})^{2n}, \exists Z_2 \in (\mathrm{span}\{\alpha_i\})^{2n}$ *with* $Z = Z_1 + Z_2$ *such that* $SZ_1 = Y$ *and* $SZ_2 = 0$.

Proof. It follows from linearly independence of $\{\alpha_i\}_{i=1}^s \cup \{1, r\}$ that

$$\mathrm{span}\{1, r\} \cap \mathrm{span}\{\alpha_i\}_{i=1}^s = \{0\}. \tag{8}$$

Moreover, $(\mathrm{span}\{1, r, \alpha_1, \alpha_2, \cdots, \alpha_s\})^{2n} = (\mathrm{span}\{1, r\})^{2n} + (\mathrm{span}\{\alpha_i\})^{2n}$. Assume that there exists $Z \in (\mathrm{span}\{1, r, \alpha_1, \cdots, \alpha_s\})^{2n}$ such that $SZ = Y$. Then $\exists Z_1 \in (\mathrm{span}\{1, r\})^{2n}, \exists Z_2 \in (\mathrm{span}\{\alpha_i\})^{2n}$ with $Z = Z_1 + Z_2$ subject to $S(Z_1 + Z_2) = Y$ or $SZ_2 = Y - SZ_2$. Because $Y, Z_1 \in (\mathrm{span}\{1, r\})^{2n}$, $SZ_2 = Y - SZ_1 \in (\mathrm{span}\{1, r\})^{2n}$. But $SZ_2 \in (\mathrm{span}\{\alpha_i\})^{2n}$, it follows from Eq. (8) that $SZ_2 = 0$ and $SZ_1 = Y$. The sufficiency is a straightforward verification. □

If x is a nontriangular fuzzy strong solution of $Ax = y$ and the membership functions of all entries of x belong to $(\mathrm{span}\{1, r, \alpha_1, \cdots, \alpha_s\})$, then X derived by Eq. (5) is a solution of $SX = Y$ and $X \in (\mathrm{span}\{1, r, \alpha_1, \cdots, \alpha_s\})^{2n}$. Decomposing X to linear part X_1 and nonlinear part X_2 leads to a triangular fuzzy number solution $\hat{x}$ of $Ax = y$ whose membership functions are given by X_1 according to Eq. (5). This is a straightforward proof for Theorem 3.5. The nonlinear part X_2 is a solution of $SX = 0$ in $(\mathrm{span}\{\alpha_i\})^{2n}$. If there is not such nonlinear solution, then TFLS $Ax = y$ has not non-triangular fuzzy number solution whose membership functions are all in $(\mathrm{span}\{1, r, \alpha_1, \cdots, \alpha_s\})$.

Theorem 4.2 *Whether* $SX = 0$ *has nonzero solution in* $(\mathrm{span}\{\alpha_i\})^{2n}$ *does not depend on the choice of* $\{\alpha_i\}$*, where* $\{\alpha_i\}_{i=1}^s$ *is a group of nonlinear functions such that* $\{\alpha_i\}_{i=1}^s \cup \{1, r\}$ *is linearly independent.*

Proof. From Theorem 3.4, $Ax = y$ has not non-triangular fuzzy number solution when $\texttt{rank}S \geq 2n$. So, $SX = 0$ has only trivial solution in $(\text{span}\{\alpha_i\})^{2n}$. Now assume $\texttt{rank}S < 2n$. Let $Z \in (\text{span}\{\alpha_i\})^{2n}$ be a solution of $SX = 0$ whose entries $Z_j = \sum_{t=1}^{s} k_{jt}\alpha_t, (j = 1, \cdots, 2n)$. Accordingly,

$$Z = \begin{pmatrix} Z_1 \\ \vdots \\ Z_{2n} \end{pmatrix} = \begin{pmatrix} \sum_{t=1}^{s} k_{1t}\alpha_t \\ \vdots \\ \sum_{t=1}^{s} k_{2m,t}\alpha_t \end{pmatrix} = \begin{pmatrix} k_{11} & \cdots & k_{1s} \\ \vdots & \ldots & \vdots \\ k_{2m,1} & \cdots & k_{2m,s} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_s \end{pmatrix}, \tag{9}$$

Since $\{\alpha_i\}_{i=1}^{2n}$ are linear independent,

$$SY = 0 \Leftrightarrow S \begin{pmatrix} k_{11} & \cdots & k_{1s} \\ \vdots & \ldots & \vdots \\ k_{2m,1} & \cdots & k_{2n,s} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_s \end{pmatrix} = 0 \Leftrightarrow S \begin{pmatrix} k_{11} & \cdots & k_{1s} \\ \vdots & \ldots & \vdots \\ k_{2m,1} & \cdots & k_{2m,s} \end{pmatrix} = 0. \quad \square \tag{10}$$

This Theorem implies that if we receive a nontriangular fuzzy number solution of TFLS $Ax = y$, then replaced those nonlinear functions in memberships of this solution by other nonlinear functions which is also linearly independent of $\{1, r\}$ leads to another nontriangular fuzzy number solution of $Ax = y$. For example, changing r^2 in the solutions in Exampe 3.1 to $\sin(\pi r/2)$ gives another nontriangular fuzzy number solution, $\bar{x} = \{\bar{x}_1, \bar{x}_2\}$ to $x_1 + x_2 = (2r, 4 - 2r)$, where $\bar{x}_1 = (\sin(\pi r/2) - 1, 1 - \sin(\pi r/2)), \bar{x}_2 = (1 + 2r - \sin(\pi r/2), 3 - 2r + \sin(\pi r/2))$.

Theorem 4.3 *Functional linear system $SX = 0$ has nontrivial solution in* $(\text{span}\{\alpha_i\})^{2n}$ *if and only if* $\text{rank}S < 2n$.

Proof. It is sufficient, due to Theorem 3.4, to prove that $SX = 0$ has nontrivial solution in $(\text{span}\{\alpha_i\})^{2n}$ when $\texttt{rank}S < 2n$. In fact, the family of all solutions to $SX = 0$ in $\mathbb{R}^{2n}$ forms a linear space, written as H.

Take nonzero vectors $(k_{11}, \cdots, k_{2n,1})^T, \cdots, (k_{1s}, \cdots, k_{2m,s})^T$ from H and construct functions $Z_j = \sum_{t=1}^{s} k_{jt}\alpha_t, (j = 1, \cdots, 2n)$ and function vector $Z = (Z_1, Z_2, \cdots, Z_{2n})^T$, then $SZ = 0$ and $Z \in (\text{span}\{\alpha_i\})^{2n}$. $\square$

In the following we deal with the existence of linear function solutions to $SX = Y$ in $(\text{span}\{1, r\})^{2n}$. Let $X = (\underline{X}, -\overline{X})^T$ and $Y = (\underline{Y}, -\overline{Y})^T$. Since $X, Y \in (\text{span}\{1, r\})^{2n}$, $\underline{X}, -\overline{X}, \underline{Y}, -\overline{Y}$ can be decomposed as follows $\underline{X} = \underline{c} + \underline{d}r, \overline{X} = \overline{c} + \overline{d}r, \underline{Y} = \underline{a} + \underline{e}r, \overline{Y} = \overline{a} + \overline{e}r$, here $\underline{c}, \overline{c}, \underline{a}, \overline{a}, \underline{d}, \overline{d}, \underline{e}, \overline{e} \in \mathbb{R}^n$. If write vectors $(\underline{a}, -\overline{a})^T, (\underline{e}, -\overline{e})^T$ as M, E, respectively, then $M + rE = Y$.

Theorem 4.4 *If and only if $SX = M$ and $SX = E$ both have solutions in $\mathbb{R}^{2n}$, functional linear system $SX = Y$ has solution in* $(\text{span}\{1, r\})^{2n}$.

Proof. Functional linear system $SX = Y$ is equivalent to

$$\begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix} \begin{pmatrix} \underline{c} + \underline{d}r \\ -\overline{c} - \overline{d}r \end{pmatrix} = \begin{pmatrix} \underline{a} + \underline{e}r \\ -\overline{a} - \overline{e}r \end{pmatrix}, \tag{11}$$

or

$$\begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix} \left[\begin{pmatrix} \underline{c} \\ -\overline{c} \end{pmatrix} + \begin{pmatrix} \underline{d} \\ -\overline{d} \end{pmatrix} r \right] = \begin{pmatrix} \underline{a} \\ -\overline{a} \end{pmatrix} + \begin{pmatrix} \underline{e} \\ -\overline{e} \end{pmatrix} r. \tag{12}$$

Hence, $SX = Y$ has solutions in $(\text{span}\{1, r\})^{2n}$ if and only if

$$\begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix} \begin{pmatrix} \underline{c} \\ -\overline{c} \end{pmatrix} = \begin{pmatrix} \underline{a} \\ -\overline{a} \end{pmatrix}, \tag{13}$$

and

$$\begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix} \begin{pmatrix} \underline{d} \\ -\overline{d} \end{pmatrix} = \begin{pmatrix} \underline{e} \\ -\overline{e} \end{pmatrix}, \tag{14}$$

both have solutions in $\mathbb{R}^{2n}$, or matrix equation

$$\begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix} \Gamma = \begin{pmatrix} \underline{a} & \underline{e} \\ -\overline{a} & -\overline{e} \end{pmatrix}, \tag{15}$$

is consistent. □

Definition 4.5 *If the matrix equation (15) is consistent, then we call that TFLS $Ax = y$ is consistent, where S, X, Y are defined by Eq. (5).*

Integrating the above theorems gives the next conclusion.

Theorem 4.6 *Firstly, convert TFLS $Ax = y$ into functional linear system $SX = Y$ as Eq. (5),*

I. If $Ax = y$ is not consistent, then it has no (neither triangular nor nontriangular) fuzzy number solutions.

II. If $Ax = y$ is consistent and $\mathtt{rank} S = 2n$, then it has only triangular fuzzy number solutions.

III. If $Ax = y$ is consistent and $\mathtt{rank} S < 2n$, then it has triangular fuzzy number solution and nontriangular fuzzy number solution as well.

IV. Suppose that $Z \in (\mathrm{span}\{1, r, \alpha_1, \cdots, \alpha_s\})^{2n}$ is nonlinear function solution of $SX = Y$, where $\{1, r, \alpha_1, \cdots, \alpha_s\}$ is linearly independent. Let $\{\beta_1, \cdots, \beta_s\}$ be another group of nonlinear functions which is linearly independent of $\{1, r\}$. A new function vector Z' whose entries are obtained by replacing α_i by β_i in turn is also a nonlinear function solution to $SX = Y$.

5. MODIFYING A WEAK SOLUTION OF TFLS TO A STRONG SOLUTION

The freedom in selectivity of nonlinear functions provided by Theorem 4.6 supplies us a possibility to choose appreciate functions such that they correspond to a strong solution to TFLS.

5.1 Linearly Modifying a Weak Solution to a Strong Solution

To solve $SX = Y$ in $(\mathrm{span}\{1, r\})^{2n}$ is actually to solve two systems of simultaneous linear equations $SX = M$ and $SX = E$ in $\mathbb{R}^{2n}$ with $M + rE = Y$. Denote the solution in $\mathbb{R}^{2n}$ of $SX = M$ and $SX = E$ by C, D,resp. Then $X = C + rD$ is a solution of $SX = Y$ in $(\mathrm{span}\{1, r\})^{2n}$.

Write the solution space of $SX = 0$ in $\mathbb{R}^{2n}$ as $H = \mathrm{span}\{v_1, \cdots, v_l\}$. So, $\forall Z \in \mathrm{span}\{v_1, \cdots, v_l\}, SZ = 0$.

Solve $SX = M$ and $SX = E$ in $\mathbb{R}^{2n}$ to obtain their solution C and D, if we use Eq. (5) backwards and receive a fuzzy vector $\tilde{x}$, then $\tilde{x}$ is a strong solution to TFLS $Ax = y$. Otherwise, choose an appreciate function from $\mathrm{span}\{1, r\}$, then add it to a vector in H, to form a fuzzy strong solution of $Ax = y$.

Suppose that $G = (v_1, \cdots, v_l)^T$ whose columns are all not zero vector. If the i^{th} entry of X needs to modify in order to get a fuzzy number solution to Eq. (2), we select $v_i \in H$ and $f(r) \in \text{span}\{1, r\}$. Write $v_i = (\underline{v}_i, -\overline{v}_i)^T$ and set $Z = X + v_i f(r)$. Then $\underline{Z} = \underline{X} + \underline{v}_i f(r)$, $\overline{Z} = \overline{X} + \overline{v}_i f(r)$.

To obtain a fuzzy number solution of Eq. (2), it is sufficient that

$$\begin{cases} \frac{\mathrm{d}\underline{Z}(r)}{\mathrm{d}r} = \frac{\mathrm{d}\underline{X}(r)}{\mathrm{d}r} + \underline{v}_i \frac{\mathrm{d}f(r)}{\mathrm{d}r} \geq 0, \\ \frac{\mathrm{d}\overline{Z}(r)}{\mathrm{d}r} = \frac{\mathrm{d}\overline{X}(r)}{\mathrm{d}r} + \overline{v}_i \frac{\mathrm{d}f(r)}{\mathrm{d}r} \leq 0, \\ \underline{Z}(1) \leq \overline{Z}(1), \end{cases} \tag{16}$$

Since $f(r) \in \text{span}\{1, r\}$, denote $f(r) = ar + b$, where $a, b \in \mathbb{R}$. So, Eq. (16) is equivalent to

$$\begin{cases} \underline{X}'(r) + a\underline{v}_i \geq 0, \\ \overline{X}'(r) + a\overline{v}_i \leq 0, \\ \underline{X}(1) + (a+b)\underline{v}_i \leq \overline{X}(1) + (a+b)\overline{v}_i. \end{cases} \tag{17}$$

The points satisfying the first and second inequalities in Eq. (17) consist of a convex field. It is easy to find a feasible a to fit the two inequalities or to assert its non-existence. We next give an example to illustrate the procedure.

Example 3. Consider the 2×3 TFLS[6]

$$\begin{cases} \tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 = (r, 2-r), \\ \tilde{x}_1 + \tilde{x}_2 - \tilde{x}_3 = (1+r, 3-r). \end{cases} \tag{18}$$

The coefficients matrix

$$S = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}. \tag{19}$$

This is a consistent system, and $\texttt{rank}S = 3, Y = (r, 1+r, r-2, r-3)^T$. The solution space of linear system $SX = 0$ in $\mathbb{R}^6$, $H = \text{span}\{v_1, v_2, v_3\}$, where $v_1 = (-1, 0, 1, -1, 0, 1)^T, v_2 = (0, 0, 0, -1, 1, 0)^T, v_3 = (-1, 1, 0, 0, 0, 0)^T$.

It is easy to get a solution of $SX = Y$ in $(\text{span}\{1, r\})^6$, $X = (-0.25 + 1.25r, 0.75 - 0.25r, -0.5, -1.75 + 0.75r, -0.75 + 0.25r, 0.5)^T$, which give a weak solution $\tilde{x}$ to $Ax = y$

$$\begin{cases} \tilde{x}_1 = (-0.25 + 1.25r, 1.75 - 0.75r), \\ \tilde{x}_2 = (0.75 - 0.25r, 0.75 - 0.25r), \\ \tilde{x}_3 = (-0.5, -0.5). \end{cases} \tag{20}$$

Since $\underline{x}_2$ is strictly monotonously decreasing, taking the structure of H in mind, let $\frac{1}{4}Sv_3(r-1) = 0$ in order to obtain a strong solution. Set $Z = X + \frac{r-1}{4}v_3$. A strong solution $\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3)^T$ to Eq. (18) is given by

$$\begin{cases} \tilde{z}_1 = (r, 1.75 - 0.75r), \\ \tilde{z}_2 = (0.5, 0.75 - 0.25r), \\ \tilde{z}_3 = (-0.5, -0.5). \end{cases} \tag{21}$$

5.2 Nonlinearly Modifying a Weak Solution to a Strong Solution

To solve $SX = Y$ in $(\text{span}\{1, r, \alpha_1, \alpha_2, \cdots, \alpha_s\})^{2n}$ is essentially to solve it in $(\text{span}\{1, r\})^{2n} + (\text{span}\{\alpha_1, \cdots, \alpha_s\})^{2n}$. In other words, we solve $SX = Y$ in $(\text{span}\{1, r\})^{2n}$ and solve $SX = 0$ in $(\text{span}\{\alpha_1, \cdots, \alpha_s\})^{2n}$.

The meanings for C, D, H are same as the preceding subsection. Solve $SX = M$ and $SX = E$ in $\mathbb{R}^{2n}$ to obtain their solution C and D. If we use Eq. (5) backwards and do not receive a fuzzy vector $\tilde{x}$, then choose an appreciate function from span$\{\alpha_1, \cdots, \alpha_s\}$, then add it to a vector in H, in order to construct a strong solution of $Ax = y$.

The nonlinearly modifying procedure is similar to the linear one except $f(r) \in$ span$\{\alpha_1, \cdots, \alpha_s\}$. Assume that $\alpha_i(r)$ is differentiable in $[0, 1]$. To obtain a fuzzy number solution of Eq. (2), it follows from the definition of fuzzy number that

$$\begin{cases} \frac{\mathrm{d}\underline{Z}(r)}{\mathrm{d}r} = \frac{\mathrm{d}\underline{X}(r)}{\mathrm{d}r} + \underline{v}_i \frac{\mathrm{d}f(r)}{\mathrm{d}r} \geq 0, \\ \frac{\mathrm{d}\overline{Z}(r)}{\mathrm{d}r} = \frac{\mathrm{d}\overline{X}(r)}{\mathrm{d}r} + \overline{v}_i \frac{\mathrm{d}f(r)}{\mathrm{d}r} \leq 0, \\ \underline{X}(1) + \underline{v}_i f(1) \leq \overline{X}(1) + \overline{v}_i f(1). \end{cases} \tag{22}$$

Example 4. This continues Example 3. We add a monotonously increasing function $f(r), r \in [0, 1]$ to $\underline{x}_2$ such that the resulting is increasing. Due to the structure of H, let $Z = X + v_3 f(r)$ whose entries are

$$\begin{cases} Z_1 = -0.25 + 1.25r - f(r), \\ Z_2 = 0.75 - 0.25r + f(r), \\ Z_3 = X_3, Z_4 = X_4, Z_5 = X_5, Z_6 = X_6, \end{cases} \tag{23}$$

subject to the definition of fuzzy number, i.e.

$$\begin{cases} -0.25 + 1.25r - f(r) \leq 1.75 - 0.75r, \\ 0.75 - 0.25r + f(r) \leq 0.75 - 0.25r, \quad r \in [0, 1], \end{cases} \tag{24}$$

and $-0.25 + 1.25r - f(r) \uparrow [0, 1], 0.75 - 0.25r + f(r) \uparrow [0, 1]$. Assume that f is differentiable on $[0, 1]$, then

$$\begin{cases} f(r) \leq 0, \\ f(r) \geq -2 + 2r, \quad r \in [0, 1], \end{cases} \text{and} \begin{cases} f'(r) \leq 1.25, \\ f'(r) \geq 0.25, \quad r \in [0, 1]. \end{cases}$$

Given and substituted any function $f(r)$ that satisfies the above conditions, say $f(r) = \sin r - \sin 1$, into $Z = v_3 f(r)$, Z is a solution of $SZ = Y$, say $Z = (-0.25 + 1.25r - \sin r + \sin 1, 0.75 - 0.25r + \sin r - \sin 1, -0.5, -1.75 + 0.75r, -0.75 + 0.25r, 0.5)^T$ which corresponds to a strong fuzzy number solution of TFLS (18) by Eq. (5).

6. CONCLUSIONS

In this work, we study the structure of solution space of functional linear system $SX = Y$ corresponding to fuzzy linear system $Ax = y$. Due to the structure an approach of (both linearly and nonlinearly) modifying a weak solution of triangular fuzzy linear system to a strong one is given.

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