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New Exact Solutions to the Generalized Zakharov Equations and the Complex Coupled KdV Equations

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Abstract: In this paper, we obtain several types of exact traveling wave solutions of the generalized Zakharov equations and the complex coupled KdV equations by using improved Riccati equations method. These explicit exact solutions contain solitary wave solutions, periodic wave solutions and the combined formal solitary wave solutions. The method can also be applied to solve more nonlinear partial differential equations.

Key Words: Improved Riccati equations method; Generalized Zakharov equations; Complex coupled KdV equations; Solitary wave solutions; Periodic wave solutions

1. INTRODUCTION

In recent years, the investigation of the traveling wave solutions for nonlinear evolution equations (NNEs) plays an important role in the study of nonlinear physical phenomena. These solutions can described various phenomena in physics and other fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, hydrodynamics, meteorology, life sciences and so on. In the past few decades, many significant methods have been developed, such as homogeneous balance method[1], Jacobi elliptic function method[2], Darboux transformation[3], extended F-expansion method[4], Exp-function method[5], the multiple exp-function method[6], extended tanh method[7], Bäcklund transformation[8] and the transformed rational function method[9]. In Ref. [10], Lu, etc., developed improved Riccati equations method. This method is by constructing two simplified Riccati equations which are more general and simpler than the method in [11, 12], based on this method several new families of exact solutions to some NEEs are obtained.

Then we consider the generalized Zakharov equations (GZEs) for the complex envelope $\Psi(x,t)$ of the high-frequency wave and the real low-frequency field $v(x,t)$in the form[5]

$$\begin{cases} i\Psi_t + \Psi_{xx} - 2\alpha|\Psi|^2\Psi + 2\Psi v = 0, \\ v_{tt} - v_{xx} + (|\Psi|^2)_{xx} = 0, \end{cases} \tag{1}$$

where the cubic term in first equation of Eq. (1) describes the nonlinear-self interaction in the high frequency subsystem, such a term corresponds to a self-focusing effect in plasma physics. The coefficient α is a real constant that can be a positive or negative number. Some exact solutions of system (2) were obtained in [5, 13–15].

Moreover, we will also consider the complex coupled KdV (CCKdV) equations[16]

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$$\begin{cases} u_t = \frac{1}{2}u_{xxx} - 3uu_x + 3(|v|^2)_x, \\ v_t = -v_{xxx} + 3uv_x. \end{cases} \tag{2}$$

With $v = 0$ and $Im(v) = 0$, Eq. (2) reduces respectively to the KdV equation and the Hirota-Satsuma[17, 18]. Some exact solutions of system (2) were obtained in [16, 19–21].

In this paper, we will use improved Riccati equations method to obtain several types of exact traveling wave solutions of Eq. (1) and Eq. (2) respectively, which contain solitary wave solutions, periodic wave solutions and the combined formal solitary wave solutions.

The rest of this paper is arranged as follows. In Section 2, we describe the key idea of the improved Riccati equations method. In Section 3 and Section 4, the improved Riccati equations method is applied to the GZEs (1) and the CCKdV equations (2) respectively, and obtain some new exact solutions. We conclude this paper in Section 5.

2. THE IMPROVED RICCATI EQUATIONS METHOD

In this section, we recall the improved Riccati equations method given in [10].

For a given nonlinear evolution equations with independent variables x and t

$$P(u, u_t, u_x, u_{xx}, ...) = 0, \tag{3}$$

where P is in general a polynomial in u and its various partial derivatives. Seeking its traveling wave solution of Eq. (3) by using the traveling wave transformation

$$u(x,t) = u(\xi), \quad \xi = kx + lt + \xi_0, \tag{4}$$

where k, l are constants to be determined later, ξ_0 is an arbitrary constant. Substituting (4) into (3) yields a nonlinear ordinary differential equations (ODEs)

$$O(u, u', u'', u''', ...) = 0, \tag{5}$$

where " $'$ " denotes $\frac{\mathrm{d}}{\mathrm{d}\xi}$. The next crucial step is that solution we are looking for is expressed in the general form

$$u(\xi) = \sum_{i=0}^{n} A_i f^i(\xi) + \sum_{j=1}^{n} B_j f^{j-1}(\xi) g^j(\xi), \tag{6}$$

where A_0, A_i, B_j $(i, j = 1, 2, ..., n)$, are constants to be determined later. The new variables $f(\xi)$, $g(\xi)$ satisfy the following improved Riccati equations:
Case 1

$$f'(\xi) = -qf(\xi)g(\xi), \quad g'(\xi) = q[1 - g^2(\xi) - rf(\xi)], \quad g^2(\xi) = 1 - 2rf(\xi) + (r^2 + \varepsilon)f^2(\xi), \tag{7}$$

where $\varepsilon = \pm 1$, $q \neq 0$, r are arbitrary constants. It is easy to see that Eq. (7) admits the following solutions:

$$f_1(\xi) = \frac{a}{b\cosh(q\xi) + c\sinh(q\xi) + ar}, \quad g_1(\xi) = \frac{b\sinh(q\xi) + c\cosh(q\xi)}{b\cosh(q\xi) + c\sinh(q\xi) + ar}, \tag{8}$$

when $\varepsilon = 1$: a, b, c satisfies $c^2 = a^2 + b^2$. when $\varepsilon = -1$: a, b, c satisfies $b^2 = a^2 + c^2$.
Case 2

$$f'(\xi) = qf(\xi)g(\xi), \quad g'(\xi) = q[1 + g^2(\xi) - rf(\xi)], \quad g^2(\xi) = -1 + 2rf(\xi) + (1 - r^2)f^2(\xi). \tag{9}$$

It is easy to see that Eq. (9) admits the following solutions:

$$f_2(\xi) = \frac{a}{b\cos(q\xi) + c\sin(q\xi) + ar}, \quad g_2(\xi) = \frac{b\sin(q\xi) - c\cos(q\xi)}{b\cos(q\xi) + c\sin(q\xi) + ar}, \tag{10}$$

where a, b, c satisfies $a^2 = b^2 + c^2$.

To determine u explicitly, we take the following four steps:
Step (1): Determine the integer n by balancing the highest order derivative terms with the nonlinear terms in Eq. (3) or (5) (n is usually a positive integer). If n is a fraction or a negative integer, we make the following transformation: (a) when $n = d/c$ is a fraction, we take $u(\xi) = v^{d/c}(\xi)$, then return to determine the balance constant n again; (b) when n is a negative integer, we let $u(\xi) = v^n(\xi)$, then return to determine the balance constant n again.
Step (2): Substitute (7) along with (6) and (9) along with (6) into Eq. (5) separately yields a set of algebraic equations for $f^i(\xi)g^j(\xi)$ $(i = 1, 2, ..., j = 0, 1)$. Setting the coefficients of $f^i(\xi)g^j(\xi)$ to zero derives a set of over-determined algebraic equations for A_0, A_i, B_j $(i, j = 1, 2, ..., n)$ and k, l.
Step (3): Solve the system of over-determined algebraic equations obtained in Step 2 using the Maple Package or Mathematica.
Step (4): Substitute A_0, A_i, B_j $(i, j = 1, 2, ..., n)$ and k, l, which are obtained in Step (3), to Eq. (4), Eq. (6), Eq. (8) and Eq. (10) respectively. Then we obtain many exact solutions of Eq. (3).

3. EXACT SOLUTIONS OF THE GENERALIZED ZAKHAROV EQUATIONS

In this Sections, we will consider the GZEs (1).

We take a plane wave transformation in the form

$$\Psi(x,t) = u(\xi)e^{i\xi}, \quad v(x,t) = v(\xi), \quad \xi = kx + lt + \xi_0, \tag{11}$$

where $u(\xi)$ is a real function. Substituting (11) into Eq. (1), we obtain the following ordinary differential equations

$$\begin{cases} k^2u'' - lu - k^2u - 2\alpha u^3 + 2uv = 0, \\ lu' + 2k^2u' = 0, \\ l^2v'' - k^2v'' + 2k^2u'^2 + 2k^2uu'' = 0, \end{cases} \tag{12}$$

where u, v satisfy Eq. (6) respectively. According to the idea of Step (1) in Section 2, we consider the homogeneous balance between u'' and uv in the first equation of Eq. (12) and between v'' and uu'' in the third equation of Eq. (12) respectively, moreover, by using Eq. (6), we obtain

$$\begin{cases} u(\xi) = a_0 + a_1f(\xi) + a_2g(\xi), \\ v(\xi) = b_0 + b_1f(\xi) + b_2f^2(\xi) + b_3g(\xi) + b_4f(\xi)g^2(\xi), \end{cases} \tag{13}$$

where $a_i, b_j (i = 0, 1, 2; j = 0, 1, 2, 3, 4)$ are constants to be determined later, and f, g satisfy Eq. (7) or Eq. (9) respectively.
State 1. Substituting (13) with (7) into (12), the left hand side of Eq. (12) is converted into a polynomial of $f^i(\xi)g^j(\xi)$ $(i = 0, 1, ..., 5; j = 0, 1)$, then setting each coefficients to zero, we get a set of over-determined algebraic system with respect to the unknown a_i, b_j $(i = 0, 1, 2; j = 0, 1, 2, 3, 4), k, l$. Solving the system of over-determined algebraic equations using Maple Package, we can distinguish two cases namely:

Case 1

$$\begin{aligned}&\varepsilon=\pm 1,\quad r=0,\quad a_0=a_1=b_1=b_3=b_4=0,\quad l=-2\,k^2,\quad b_2=-\frac{q^2\,k^2\,\varepsilon}{1-\alpha+4k^2\alpha},\\&a_2=\pm\frac{kq\,\sqrt{(1-\alpha+4k^2\alpha)\,(-1+4\,k^2)}}{1-\alpha+4k^2\alpha},\quad b_0=\frac{k^2\,(-1+\alpha-4\,k^2\,\alpha+8\,\alpha\,q^2\,k^2-2\,\alpha\,q^2)}{2\,(1-\alpha+4k^2\alpha)},\end{aligned}\tag{14}$$

where $\alpha, q \neq 0, k \neq 0$ are arbitrary constants. So do the following situations.

According to Eqs. (8), (11), (13), (14), we obtain solitary wave solutions of Eq. (1) as follows:
Family 1

$$\Psi(x,t)=\frac{A_1 b\,\sinh(q(kx-2k^2t+\xi_0))+A_1 c\,\cosh(q(kx-2k^2t+\xi_0))}{b\,\cosh(q(kx-2k^2t+\xi_0))+c\,\sinh(q(kx-2k^2t+\xi_0))}e^{\mathrm{i}(kx-2k^2t+\xi_0)},\tag{15}$$

$$v(x,t)=A_2-\frac{a^2q^2\,k^2\,\varepsilon}{(1-\alpha+4k^2\alpha)(b\,\cosh(q(kx-2k^2t+\xi_0))+c\,\sinh(q(kx-2k^2t+\xi_0)))^2},\tag{16}$$

where $A_1=\pm\frac{kq\,\sqrt{(1-\alpha+4k^2\alpha)\,(-1+4\,k^2)}}{1-\alpha+4k^2\alpha}$, $A_2=\frac{k^2\,(-1+\alpha-4\,k^2\,\alpha+8\,\alpha\,q^2\,k^2-2\,\alpha\,q^2)}{2\,(1-\alpha+4k^2\alpha)}$, and when $\varepsilon=1$: a, b, c satisfies $c^2=a^2+b^2$, when $\varepsilon=-1$: a, b, c satisfies $b^2=a^2+c^2$.
Case 2

$$\begin{aligned}&\varepsilon=-1,\quad r=\pm 1,\quad a_0=a_1=b_3=0,\quad l=-2k^2,\quad b_2=\pm 2b_4,\\&b_0=\frac{k^2(-2+2\alpha-8k^2\alpha+4\alpha q^2k^2-\alpha q^2)}{4(1-\alpha+4k^2\alpha)},\quad b_1=\frac{\pm k^2q^2-2b_4+2\alpha b_4-8k^2\alpha b_4}{2(1-\alpha+4k^2\alpha)},\\&a_2=\pm\frac{\sqrt{(1-\alpha+4k^2\alpha)(-1+4\,k^2)}}{2(1-\alpha+4k^2\alpha)}kq,\end{aligned}\tag{17}$$

where $\alpha, q \neq 0, k \neq 0, b_4$ are arbitrary constants. So do the following situations.

According to Eqs. (8), (11), (13), (17), we obtain solitary wave solutions of Eq. (1) as follows:
Family 2

$$\Psi(x,t)=\frac{A_3(b\,\sinh(q\xi)+c\,\cosh(q\xi))}{b\,\cosh(q\xi)+c\,\sinh(q\xi)\pm a}e^{\mathrm{i}\xi},\tag{18}$$

$$\begin{aligned}v(x,t)=A_4&+\frac{A_5 a}{b\,\cosh(q\xi)+c\,\sinh(q\xi)\pm a}\pm\frac{2b_4a^2}{(b\,\cosh(q\xi)+c\,\sinh(q\xi)\pm a)^2}\\&+\frac{b_4a}{b\,\cosh(q\xi)+c\,\sinh(q\xi)\pm a}\left(\frac{b\,\sinh(q\xi)+c\,\cosh(q\xi)}{b\,\cosh(q\xi)+c\,\sinh(q\xi)\pm a}\right)^2,\end{aligned}\tag{19}$$

where $A_3=\pm\frac{\sqrt{(1-\alpha+4k^2\alpha)\,(-1+4\,k^2)}}{2\,(1-\alpha+4k^2\alpha)}kq$, $A_4=\frac{k^2\,(-2+2\,\alpha-8\,k^2\,\alpha+4\,\alpha\,q^2\,k^2-\alpha\,q^2)}{4\,(1-\alpha+4k^2\alpha)}$, $A_5=\frac{\pm k^2\,q^2-2\,b_4+2\,\alpha\,b_4-8\,k^2\,\alpha\,b_4}{2(1-\alpha+4k^2\alpha)}$, $\xi=kx-2k^2t+\xi_0$, b_4 is the arbitrary constant, and a, b, c satisfies $b^2=a^2+c^2$.
State 2. In common, substituting (13) with (9) into (12), the left hand side of Eq. (12) we can obtain determine the following solutions:
Case 3

$$\begin{aligned}&r=0,\quad a_0=a_2=b_1=b_3=b_4=0,\quad l=-2\,k^2,\quad b_0=\frac{k^2\,(q^2-1)}{2},\\&a_1=\pm\frac{qk\,\sqrt{(-\alpha+4\alpha k^2+1)\,(-1+4\,k^2)}}{-\alpha+4\alpha k^2+1},\quad b_2=-\frac{k^2\,q^2}{-\alpha+4\alpha k^2+1},\end{aligned}\tag{20}$$

where $\alpha, q \neq 0, k \neq 0$ are arbitrary constants. So do the following situations.

According to Eqs. (10), (11), (13), (20), we obtain solitary wave solutions of Eq. (1) as follows:

Family 3

$$\Psi(x,t) = \frac{aA_6}{b\cos(q(kx-2k^2t+\xi_0)) + c\sin(q(kx-2k^2t+\xi_0))} e^{\mathrm{i}(kx-2k^2t+\xi_0)}, \tag{21}$$

$$v(x,t) = \frac{k^2(q^2-1)}{2} - \frac{a^2k^2q^2}{(-\alpha+4\alpha k^2+1)(b\cos(q(kx-2k^2t+\xi_0))+c\sin(q(kx-2k^2t+\xi_0)))^2}, \tag{22}$$

where $A_6 = \pm\frac{qk\sqrt{(-\alpha+4\alpha k^2+1)(-1+4k^2)}}{-\alpha+4\alpha k^2+1}$, and a, b, c satisfies $a^2 = b^2 + c^2$.

Case 4

$$\begin{aligned} &r=\pm 1,\quad a_0=a_1=b_3=0,\quad l=-2k^2,\quad a_2=\pm\frac{qk\sqrt{(-\alpha+4\alpha k^2+1)(-1+4k^2)}}{2(-\alpha+4\alpha k^2+1)},\\ &b_0=-\frac{k^2(-2\alpha+8\alpha k^2+2+4k^2q^2\alpha-q^2\alpha)}{4(-\alpha+4\alpha k^2+1)},\\ &b_1=-\frac{2\alpha b_4-8\alpha k^2b_4\pm k^2q^2-2b_4}{2(-\alpha+4\alpha k^2+1)},\quad b_2=\mp 2b_4, \end{aligned} \tag{23}$$

where $\alpha, q \neq 0, k \neq 0, b_4$ are arbitrary constants. So do the following situations.

According to Eqs. (10), (11), (13), (23), we obtain solitary wave solutions of Eq. (1) as follows:

Family 4

$$\Psi(x,t) = \frac{A_7(b\sin(q\xi) - c\cos(q\xi))}{b\cos(q\xi) + c\sin(q\xi) \pm a} e^{\mathrm{i}\xi}, \tag{24}$$

$$\begin{aligned} v(x,t) = A_8 &+ \frac{A_9 a}{b\cos(q\xi)+c\sin(q\xi)\pm a} \mp \frac{2b_4a^2}{(b\cos(q\xi)+c\sin(q\xi)\pm a)^2}\\ &+ \frac{b_4 a}{b\cos(q\xi)+c\sin(q\xi)\pm a}\left(\frac{b\sin(q\xi)-c\cos(q\xi)}{b\cos(q\xi)+c\sin(q\xi)\pm a}\right)^2, \end{aligned} \tag{25}$$

where $A_7 = \pm\frac{qk\sqrt{(-\alpha+4\alpha k^2+1)(-1+4k^2)}}{2(-\alpha+4\alpha k^2+1)}$, $A_8 = -\frac{k^2(-2\alpha+8\alpha k^2+2+4k^2q^2\alpha-q^2\alpha)}{4(-\alpha+4\alpha k^2+1)}$, $A_9 = -\frac{2\alpha b_4-8\alpha k^2b_4\pm k^2q^2-2b_4}{2(-\alpha+4\alpha k^2+1)}$, $\xi = kx - 2k^2t + \xi_0, b_4$ is the arbitrary constant, and a, b, c satisfies $a^2 = b^2 + c^2$.

Case 5

$$\begin{aligned} &r=0,\quad a_0=a_1=b_1=b_3=b_4=0,\quad a_2=\pm\frac{qk\sqrt{(-\alpha+4\alpha k^2+1)(-1+4k^2)}}{-\alpha+4\alpha k^2+1},\\ &b_0=-\frac{k^2(-\alpha+4\alpha k^2+1+8k^2q^2\alpha-2q^2\alpha)}{2(-\alpha+4\alpha k^2+1)},\quad b_2=-\frac{k^2q^2}{-\alpha+4\alpha k^2+1}, l=-2k^2, \end{aligned} \tag{26}$$

where $\alpha, q \neq 0, k \neq 0$. So do the following situations.

According to Eqs. (10), (11), (13), (26), we obtain solitary wave solutions of Eq. (1) as follows:

Family 5

$$\Psi(x,t) = \frac{A_{10}(b\sin(q\xi) - c\cos(q\xi))}{b\cos(q\xi) + c\sin(q\xi)} e^{\mathrm{i}\xi}, \tag{27}$$

$$v(x,t) = A_{11} - \frac{a^2k^2q^2}{(-\alpha+4\alpha k^2+1)(b\cos(q\xi)+c\sin(q\xi))^2}, \tag{28}$$

where $A_{10} = \pm\frac{qk\sqrt{(-\alpha+4\alpha k^2+1)(-1+4k^2)}}{-\alpha+4\alpha k^2+1}$, $A_{11} = -\frac{k^2(-\alpha+4\alpha k^2+1+8k^2q^2\alpha-2q^2\alpha)}{2(-\alpha+4\alpha k^2+1)}$, $\xi = kx - 2k^2t + \xi_0$, and a, b, c satisfies $a^2 = b^2 + c^2$.

Moreover, we will illuminate the evolution property of the solutions by some interesting figures of solutions. For simplicity, we will only give out figures (Fig.1, Fig.2) of Eqs. (18) and (19).

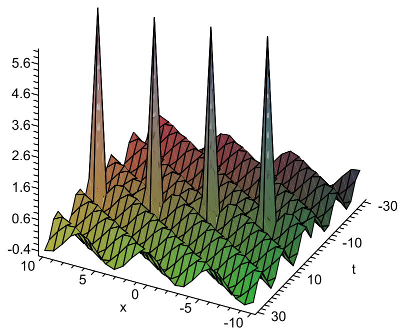

Figure 1: The single generalized solitary wave solution for the real part of Eq. (18), where $\alpha = k = a = b = q = 1, c = 0$, x is from -10 to 10, and t is from -30 to 30

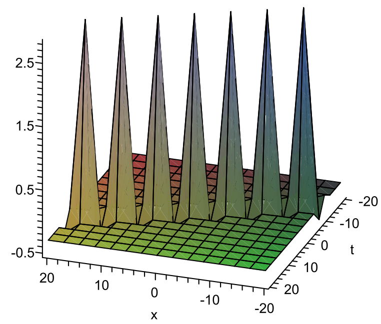

Figure 2: The single generalized solitary wave solution for Eq. (19), where $\alpha = k = a = b = q = b_4 = 1, c = 0$, x is from -20 to 20, and t is from -20 to 20

4. EXACT SOLUTIONS OF THE COMPLEX COUPLED KDV EQUATIONS

In this Sections, we will consider the CCKdV (2). Similarly, we take a plane wave transformation in the form

$$u(x,t) = u(\xi), \qquad v(x,t) = v(\xi)e^{i\xi}, \qquad \xi = kx + lt + \xi_0, \tag{29}$$

where $v(\xi)$ is a real function. Substituting (29) into Eq. (2), we obtain the following ordinary differential equations

$$\begin{cases} \frac{1}{2}k^3u''' - 3kuu' + 6kvv' - lu' = 0, \\ k^3v''' - 3k^3v' - 3kuv' + lv' = 0, \\ 3k^3v'' - k^3v - 3kuv + lv = 0, \end{cases} \tag{30}$$

where u, v satisfy Eq. (6) respectively. Similarly, we obtain

$$\begin{cases} u(\xi) = a_0 + a_1f(\xi) + a_2f^2(\xi) + a_3g(\xi) + a_4f(\xi)g^2(\xi), \\ v(\xi) = b_0 + b_1f(\xi) + b_2f^2(\xi) + b_3g(\xi) + b_4f(\xi)g^2(\xi), \end{cases} \tag{31}$$

where $a_i, b_j (i, j = 0, 1, 2, 3, 4)$ are constants to be determined later, and f, g satisfy Eq. (7) or Eq. (9) respectively.

State 1. Substituting (31) with (7) into (30), using the same process, we can distinguish two cases namely:

Case 1

$$\begin{aligned} &\varepsilon = -1, \quad r = \pm 1, \quad l = 1, \quad a_3 = b_0 = b_3 = 0, \\ &a_1 = \mp\tfrac{1}{2}a_2, \quad b_1 = -b_4, \quad a_4 = \pm\tfrac{1}{2}a_2, \quad b_2 = \pm 2b_4, \end{aligned} \tag{32}$$

where $q \neq 0, k \neq 0, a_0, a_2, b_4$ are arbitrary constants. So do the following situation.

According to Eqs. (8), (29), (31), (32), we obtain solitary wave solutions of Eq. (2) as follows:

Family 1

$$\begin{aligned} u(x,t) = (a_0 \mp &\frac{a_2a}{2b\cosh(q\xi)+2c\sinh(q\xi)\pm 2a} + \frac{a_2a^2}{(b\cosh(q\xi)+c\sinh(q\xi)\pm a)^2} \\ &\pm\frac{a_2a}{2(b\cosh(q\xi)+c\sinh(q\xi)\pm a)}\left(\frac{b\sinh(q\xi)+c\cosh(q\xi)}{b\cosh(q\xi)+c\sinh(q\xi)\pm a}\right)^2)e^{i\xi}, \end{aligned} \tag{33}$$

$$\begin{aligned} v(x,t) = &-\frac{b_4 a}{b\cosh(q\xi)+c\sinh(q\xi)\pm a} \pm \frac{2b_4 a^2}{(b\cosh(q\xi)+c\sinh(q\xi)\pm a)^2} \\ &+\frac{b_4 a}{b\cosh(q\xi)+c\sinh(q\xi)\pm a}\left(\frac{b\sinh(q\xi)+c\cosh(q\xi)}{b\cosh(q\xi)+c\sinh(q\xi)\pm a}\right)^2, \end{aligned} \quad (34)$$

where $\xi = kx + t + \xi_0$, and a, b, c satisfies $b^2 = a^2 + c^2$.

Case 2

$$\begin{aligned} &\varepsilon = -1, \quad r = \pm 1, \quad l = \tfrac{1}{2}q^2k^3 - 3a_0k, \quad a_3 = b_0 = b_3 = 0, \\ &a_2 = -2k^2q^2 \mp \tfrac{1}{2}a_1, \quad b_1 = -b_4, \quad a_4 = \mp k^2q^2 - a_1, \quad b_2 = \pm 2b_4, \end{aligned} \quad (35)$$

where $q \neq 0, k \neq 0, a_0, a_1, b_4$ are arbitrary constants. So do the following situation.

According to Eqs. (8), (29), (31), (35), we obtain solitary wave solutions of Eq. (2) as follows:

Family 2

$$\begin{aligned} u(x,t) = &(a_0 + \frac{a_1 a}{b\cosh(q\xi)+c\sinh(q\xi)\pm a} - \frac{(2k^2q^2 \pm \frac{1}{2}a_1)a^2}{(b\cosh(q\xi)+c\sinh(q\xi)\pm a)^2} \\ &-\frac{(a_1 \pm k^2q^2)a}{b\cosh(q\xi)+c\sinh(q\xi)\pm a}\left(\frac{b\sinh(q\xi)+c\cosh(q\xi)}{b\cosh(q\xi)+c\sinh(q\xi)\pm a}\right)^2)e^{i\xi}, \end{aligned} \quad (36)$$

$$\begin{aligned} v(x,t) = &-\frac{b_4 a}{b\cosh(q\xi)+c\sinh(q\xi)\pm a} \pm \frac{2b_4 a^2}{(b\cosh(q\xi)+c\sinh(q\xi)\pm a)^2} \\ &+\frac{b_4 a}{b\cosh(q\xi)+c\sinh(q\xi)\pm a}\left(\frac{b\sinh(q\xi)+c\cosh(q\xi)}{b\cosh(q\xi)+c\sinh(q\xi)\pm a}\right)^2, \end{aligned} \quad (37)$$

where $\xi = kx + (\frac{1}{2}q^2k^3 - 3a_0k)t + \xi_0$, and a, b, c satisfies $b^2 = a^2 + c^2$.

Case 3

$$\begin{aligned} &\varepsilon = -1, \quad r = \pm 1, \quad l = k^3 + 3a_0k, \quad a_3 = b_3 = 0, \\ &a_2 = \pm 2a_4, \quad b_1 = -b_4, \quad b_2 = \pm 2b_4, \quad a_1 = -a_4 \end{aligned} \quad (38)$$

where $q \neq 0, k \neq 0, a_0, a_4, b_0, b_4$ are arbitrary constants. So do the following situation.

According to Eqs. (8), (29), (31), (38), we obtain solitary wave solutions of Eq. (2) as follows:

Family 3

$$\begin{aligned} u(x,t) = &(a_0 - \frac{a_4 a}{b\cosh(q\xi)+c\sinh(q\xi)\pm a} \pm \frac{2a_4 a^2}{(b\cosh(q\xi)+c\sinh(q\xi)\pm a)^2} \\ &+\frac{a_4 a}{b\cosh(q\xi)+c\sinh(q\xi)\pm a}\left(\frac{b\sinh(q\xi)+c\cosh(q\xi)}{b\cosh(q\xi)+c\sinh(q\xi)\pm a}\right)^2)e^{i\xi}, \end{aligned} \quad (39)$$

$$\begin{aligned} v(x,t) = b_0 &- \frac{b_4 a}{b\cosh(q\xi)+c\sinh(q\xi)\pm a} \pm \frac{2b_4 a^2}{(b\cosh(q\xi)+c\sinh(q\xi)\pm a)^2} \\ &+\frac{b_4 a}{b\cosh(q\xi)+c\sinh(q\xi)\pm a}\left(\frac{b\sinh(q\xi)+c\cosh(q\xi)}{b\cosh(q\xi)+c\sinh(q\xi)\pm a}\right)^2, \end{aligned} \quad (40)$$

where $\xi = kx + (k^3 + 3a_0k)t + \xi_0$, and a, b, c satisfies $b^2 = a^2 + c^2$.

State 2. Substituting (31) with (9) into (30), using the same process, we can distinguish two cases namely:

Case 4

$$\begin{aligned} &r = \pm 1, \quad a_2 = a_3 = a_4 = b_0 = b_3 = 0, \\ &l = -\tfrac{1}{2}q^2k^3 - 3ka_0, \quad a_1 = \pm k^2q^2, \quad b_1 = b_4, \quad b_2 = \mp 2b_4, \end{aligned} \quad (41)$$

where $q \neq 0, k \neq 0, a_0, b_4$ are arbitrary constants. So do the following situation.

According to Eqs. (10), (29), (31), (41), we obtain solitary wave solutions of Eq. (2) as follows:

Family 4

$$u(x,t) = (a_0 \pm \frac{k^2q^2 a}{b\cos(q\xi)+c\sin(q\xi)\pm a})e^{i\xi}, \quad (42)$$

$$\begin{aligned} v(x,t) = &\frac{b_4 a}{b\cos(q\xi)+c\sin(q\xi)\pm a} \mp \frac{2b_4 a^2}{(b\cos(q\xi)+c\sin(q\xi)\pm a)^2} \\ &+\frac{b_4 a}{b\cos(q\xi)+c\sin(q\xi)\pm a}\left(\frac{b\sin(q\xi)-c\cos(q\xi)}{b\cos(q\xi)+c\sin(q\xi)\pm a}\right)^2, \end{aligned} \quad (43)$$

where $\xi = kx - (\frac{1}{2}q^2k^3 + 3a_0k)t + \xi_0$, and a, b, c satisfies $a^2 = b^2 + c^2$.

Case 5

$$\begin{aligned} &r = \pm 1, \quad l = 3ka_0 + k^3, \quad a_3 = b_3 = 0, \\ &a_2 = \mp 2a_1, \quad a_4 = a_1, \quad b_2 = \mp 2b_1, \quad b_4 = b_1, \end{aligned} \quad (44)$$

where $q \neq 0, k \neq 0, a_0, a_1, b_0, b_1$ are arbitrary constants. So do the following situation.

According to Eqs. (10), (29), (31), (44), we obtain solitary wave solutions of Eq. (2) as follows:

Family 5

$$\begin{aligned} u(x,t) = (a_0 &+ \frac{a_1 a}{b\cos(q\xi)+c\sin(q\xi)\pm a} \mp \frac{2a_1 a^2}{(b\cos(q\xi)+c\sin(q\xi)\pm a)^2} \\ &+ \frac{a_1 a}{b\cos(q\xi)+c\sin(q\xi)\pm a}\left(\frac{b\sin(q\xi)-c\cos(q\xi)}{b\cos(q\xi)+c\sin(q\xi)\pm a}\right)^2)e^{i\xi}, \end{aligned} \tag{45}$$

$$\begin{aligned} v(x,t) = b_0 &+ \frac{b_1 a}{b\cos(q\xi)+c\sin(q\xi)\pm a} \mp \frac{2b_1 a^2}{(b\cos(q\xi)+c\sin(q\xi)\pm a)^2} \\ &+ \frac{b_1 a}{b\cos(q\xi)+c\sin(q\xi)\pm a}\left(\frac{b\sin(q\xi)-c\cos(q\xi)}{b\cos(q\xi)+c\sin(q\xi)\pm a}\right)^2, \end{aligned} \tag{46}$$

where $\xi = kx + (3ka_0 + k^3)t + \xi_0$, and a, b, c satisfies $a^2 = b^2 + c^2$.

Case 6

$$\begin{aligned} &r = \pm 1, \quad l = -\tfrac{1}{2}q^2k^3 - 3ka_0, \quad a_3 = b_0 = b_3 = 0, \\ &a_2 = 2k^2q^2 \mp 2a_1, \quad a_4 = \mp k^2q^2 + a_1, \quad b_2 = \mp 2b_1, \quad b_4 = b_1, \end{aligned} \tag{47}$$

where $q \neq 0, k \neq 0, a_0, a_1, b_1$ are arbitrary constants. So do the following situation.

According to Eqs. (10), (29), (31), (47), we obtain solitary wave solutions of Eq. (2) as follows:

Family 6

$$\begin{aligned} u(x,t) = (a_0 &+ \frac{a_1 a}{b\cos(q\xi)+c\sin(q\xi)\pm a} + \frac{(2k^2q^2 \mp 2a_1)a^2}{(b\cos(q\xi)+c\sin(q\xi)\pm a)^2} \\ &\mp \frac{(k^2q^2+a_1)a}{b\cos(q\xi)+c\sin(q\xi)\pm a}\left(\frac{b\sin(q\xi)-c\cos(q\xi)}{b\cos(q\xi)+c\sin(q\xi)\pm a}\right)^2)e^{i\xi}, \end{aligned} \tag{48}$$

$$\begin{aligned} v(x,t) = &\frac{b_1 a}{b\cos(q\xi)+c\sin(q\xi)\pm a} \mp \frac{2b_1 a^2}{(b\cos(q\xi)+c\sin(q\xi)\pm a)^2} \\ &+ \frac{b_1 a}{b\cos(q\xi)+c\sin(q\xi)\pm a}\left(\frac{b\sin(q\xi)-c\cos(q\xi)}{b\cos(q\xi)+c\sin(q\xi)\pm a}\right)^2, \end{aligned} \tag{49}$$

where $\xi = kx - (\frac{1}{2}q^2k^3 + 3a_0k)t + \xi_0$, and a, b, c satisfies $a^2 = b^2 + c^2$.

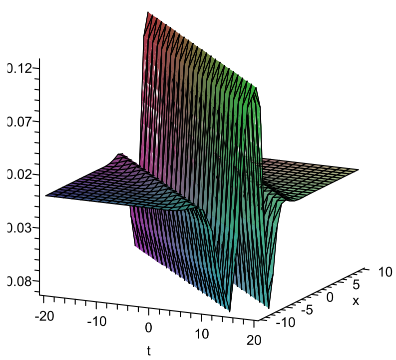

Figure 3: The single generalized solitary wave solution for the real part of Eq. (36), where $a_0 = c = 0, a_1 = a = b = k = q = 1$, x is from -10 to 10, t is from -20 to 20

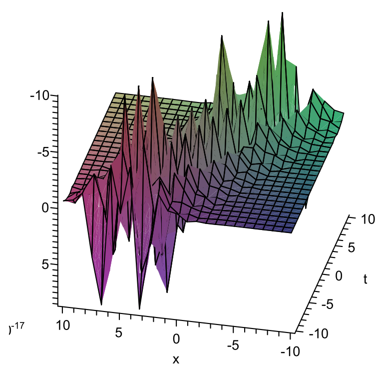

Figure 4: The single generalized solitary wave solution for Eq. (37), where $a_0 = c = 0$, $a_1 = a = b = k = q = b_4 = 1$, x is from -10 to 10, and t is from -10 to 10

Similarly, we will illuminate the evolution property of the solutions by some interesting figures of solutions. For simplicity, we will only give out figures (Fig.3, Fig.4) of Eqs. (36) and (37).

5. CONCLUSIONS

In this study, we have used the improved Riccati equations method successfully to derive exact solutions of the generalized-Zakharov equations and the complex coupled KdV equations. And we have obtained many exact solutions, including solitary wave solutions, periodic wave solutions and the combined formal solitary wave solutions.

In Ref. [5, 22–24], some exact solutions of the generalized-Zakharov equations have been obtained by using the Exp-function method, a new rational auxiliary equation method, a variational iteration method and in Ref. [21], some exact solutions of the complex coupled KdV equations have been obtained by using a generalized tanh method. In this paper, the obtained exact solutions include some solutions in Ref. [5, 21–24], but we also give many new exact solutions, such as Eq. (27), Eq. (28) and Eq. (48), Eq. (49). The solution procedure presented in this paper is available for other NEEs. It is also shown that the improved Riccati equations method provides a effective and powerfully mathematical tool for obtaining exact solutions of nonlinear partial differential equations.

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