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# Related Distribution of Prime Numbers

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**Abstract:** The surplus model is established, and the distribution of prime numbers are solved by using the surplus model.

**Key words:** Surplus model; Distribution of prime numbers

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## 1. SURPLUS MODEL

In the range of $1 \sim N$, $r_1$ represents the quantity of numbers which do not contain a factor 2, $r_1 = [N \cdot \frac{1}{2}]$ ($[x]$ is the largest integer not greater than $x$); $r_2$ represents the quantity of numbers which do not contain factor 3, $r_2 = [r_1 \cdot \frac{2}{3}]$; $r_3$ represents the quantity of numbers do not contain factor 5, $r_3 = [r_2 \cdot \frac{4}{5}]$; ...

In the range of $1 \sim N$, $r_i$ represents the quantity of numbers which do not contain factor $p_i$ , the method of the surplus numbers $r_{i-1}$ to $r_i$ is called *the residual model of related distribution of prime numbers.*

In the range of $1 \sim N$, excluding all multiples of prime numbers which are not greater than $\sqrt{M}$ $(M \leq N)$, then $r$ is the surplus number. For the factor that $[\frac{rM}{N}] \leq r$, so there are at least $[\frac{rM}{N}]$ related primes in the range os $M$.

## 2. RELATED DISTRIBUTION OF PRIME NUMBERS

### *2.1. Distribution of Prime Numbers*

Exclusion the multiples of 2, 3, 5 in natural numbers, the rest numbers can be categorized as eight numeric axes: $A_1$, $B_1$, $A_3$, $B_3$, $A_7$, $B_7$, $A_9$, $B_9$ (Axis $A$: $3n+2$, axis $B$: $3n+1$).

On number axis ($A$-axis or $B$-axis), $r_0$ represents the quantity of numbers which do not contain factors 2, 3, 5, 7, $r_0 = [\frac{8}{35} \cdot N]$; $r_1$ represents the quantity of numbers which do not contain prime factor $P_1$ ($P_1 > 7$), $r_1 = [r_0 \cdot \frac{p_1 - 1}{p_1}]$; $r_2$ represents the quantity of numbers which do not contain prime factor $P_2$, $r_2 = [r_1 \cdot \frac{p_2 - 1}{p_2}]$; ... $r_i = [r_{i-1} \cdot \frac{p_i - 1}{p_i}]$.

The method of the surplus numbers $r_{i-1}$ to $r_i$ is called *the surplus model of distribution of prime numbers.*

$S_n$ is the quantity of prime numbers in the range of $N$, make $M = p_n^2$, by

$$S_n = [r_{n-1} \cdot \frac{p_n - 1}{p_n} \cdot \frac{M}{N}] = [r_{n-1} \cdot \frac{p_n - 1}{p_n} \cdot \frac{p_n^2}{N}]. \tag{1}$$

$$S_{n-1} = [[r_{n-2} \cdot \frac{p_{n-1} - 1}{p_{n-1}}] \frac{p_{n-1}^2}{N}]. \tag{2}$$

We can obtain

$$S_n = [\frac{(p_n - 1)p_n}{p_{n-1}^2} \cdot S_{n-1}],$$

where $S_1 = [\frac{8 \times 10 \times 11}{35}] = 25$. ($r_{n-1} = [...[\frac{8}{35} \cdot N] \frac{p_1 - 1}{p_1}]...] \frac{p_{n-1} - 1}{p_{n-1}}]$).

For the fact that $\frac{(p_i - 1)p_i}{p_{i-1}^2} > 1$, so $\{[\frac{(p_i - 1)p_i}{p_{i-1}^2} \cdot S_{i-1}]\}$ $(i = 1, 2, ..., n)$ are ascending series in which all numbers are not less than 25. Therefore, $\lim_{n \to \infty} S_n = \infty$, there are infinite prime numbers in natural numbers.

### *2.2. Distribution of Primes in Arithmetic Series*

$S_{(a,r)}$ is the quantity of prime numbers which have the form $an + r((a, r) = 1)$, then

$$S(a, r) = \frac{S_n}{\varphi(a)} = [\frac{p_n(p_n - 1)}{\varphi(a) p_{n-1}^2} \cdot S_{n-1}],$$

where $S_1 = 25$. ($\varphi(a)$) is Euler function of $a$.)

$a$ is a constant, $\varphi(a)$ is a constant. There are infinite prime numbers which have the form $an + r((a, r) = 1)$.

### *2.3. Distribution of Prime Numbers Between $n^2$ and $(n+1)^2$*

Make $n^2 = p_n^2$, $(n+1)^2 = (p_n+1)^2$, then $S_n'$ is the quantity of prime numbers in the range of $(p_n+1)^2$,

$$S_n{}' = \left[\frac{(p_n+1)(p_n+1-1)}{p_n^2} \cdot S_n\right].$$

Let $f(n) = S_n{}' - S_n = \left[\frac{(p_n+1)(p_n+1-1)}{p_n^2} \cdot S_n\right] - [S_n] = [\frac{[S_n]}{p_n}]$, where $S_1 = 25$.

For $f(n) = [\frac{(p_n-1)}{p_{n-1}}[\frac{(p_{n-1}-1)}{p_{n-2}}[\frac{(p_{n-2}-1)}{p_{n-3}}[...[\frac{(p_2-1)}{p_1^2} \cdot S_1...]$, and $\frac{p_i-1}{p_{i-1}} > 1$. Then $\lim_{n\to\infty} f(n) = \infty$.

The quantity of prime numbers between $n^2$ and $(n+1)^2$ are increasing along with the increasing of $n$. In a small range, there is at least one prime number between $n^2$ and $(n+1)^2$. Therefore, there is at least one prime number between $n^2$ and $(n+1)^2$; and the quantity of the prime number are increasing along with the increasing of $n$.

### *2.4. The Interval Between Primes $p_{i-1}$ and $p_i$*

During the distribution of prime numbers,

$$S_n = [\frac{p_n}{p_{n-1}}[\frac{(p_n-1)}{p_{n-2}}[\frac{(p_{n-1}-1)}{p_{n-3}}[...[\frac{p_3-1}{p_1}[\frac{p_2-1}{p_1} \cdot S_1]...].$$

Let $\overline{p_i - p_{i-1}}$ be the average interval between two primes $p_{i-1}$ and $p_i$, then in the range of $p_n^2$,

$$\overline{p_i - p_{i-1}} = \frac{p_n^2}{S_n} = [\frac{p_n}{p_{n-1}}[\frac{p_{n-1}}{p_{n-1}-1}[...[\frac{p_2}{p_2-1}[\frac{p_1^2}{S_1}]...],\ \ \frac{p_i}{p_i-1} > 1.$$

and $\lim_{n\to\infty} \overline{p_i - p_{i-1}} = \infty$.

There is no maximum value of the average interval between primes $p_{i-1}$ and $p_i$. Therefore, the maximum value of the average interval between primes $p_{i-1}$ and $p_i$ tends to infinity.

### *2.5. Distribution of Twin Primes*

On number axis pair ($A_1 - B_3$ or $A_7 - B_9$ or $A_9 - B_1$), $r_0$ represents the quantity of numbers which do not contain factors 2, 3, 5, 7, $r_0 = [\frac{1}{14} \cdot N]$; $r_1$ represents the quantity of numbers which do not contain factor $p_1$, $r_1 = [r_0 \cdot \frac{p_1-2}{p_1}]$; $r_2$ represents the quantity of numbers which do not contain factor $p_2$, $r_2 = [r_1 \cdot \frac{p_2-2}{p_2}]$;... $r_i = [r_{i-1} \cdot \frac{p_i-2}{p_i}]$.

The method of the surplus numbers $r_{i-1}$ to $r_i$ is called *the surplus model of distribution of twin primes.*

$L_n$ is the quantity of twin primes in the range of $N$, let $M = p_n^2$, by

$$L_n = [r_{n-1} \cdot \frac{p_n - 2}{p_n} \cdot \frac{M}{N}] = [r_{n-1} \cdot \frac{p_n - 2}{p_n} \cdot \frac{p_n^2}{N}] \tag{3}$$

$$L_{n-1} = [[r_{n-2} \cdot \frac{p_{n-1} - 2}{p_{n-1}}] \frac{p_{n-1}^2}{N}]. \tag{4}$$

We can obtain that

$$L_n = [\frac{(p_n - 2)p_n}{p_{n-1}^2} \cdot L_{n-1}],$$

where $L_1 = [\frac{11 \times 9}{14}] = 7$. $(r_{n-1} = [...[\frac{1}{14}N]\frac{p_1 - 2}{p_1}]...]\frac{p_{n-1} - 2}{p_{n-1}}])$.

For the fact that $\frac{(p_i - 2)p_i}{p_{i-1}^2} > 1$, so $\{[\frac{(p_i - 2)p_i}{p_{i-1}^2} \cdot L_{i-1}]\}$ $(i = 1, 2, ..., n)$ are ascending series in which all numbers are not less than 7. Therefore, $\lim_{n\to\infty} L_n = \infty$, there are infinite twin primes in natural numbers.

## *2.6. Distribution of Twin Prime Numbers Within Ten*

There are two pairs of twin prime numbers within 10, these two pairs of twin prime numbers are called *twin prime numbers within ten.* For example, 11, 13, 17, 19 are twin prime numbers within ten.

On number axis pair $(A_1 - B_3 - A_7 - B_9)$, $r_0$ represents the quantity of numbers which do not contain factors 2, 3, 5, 7, $r_0 = \frac{1}{70} \cdot N$; $r_1$ represents the quantity of numbers which do not contain factor $p_1$, $r_1 = r_0 \cdot \frac{p_1 - 4}{p_1}$; $r_2$ represents the quantity of numbers which do not contain factor $p_2$, $r_2 = r_1 \cdot \frac{p_2 - 4}{p_2}$;... $r_i = r_{i-1} \cdot \frac{p_i - 4}{p_i}$.

The method of the surplus numbers $r_{i-1}$ to $r_i$ is called *the surplus model of distribution of twin prime numbers within 10.*

$T_n$ indicates the quantity of twin prime numbers within 10, let $M = p_n^2$, by

$$T_n = r_{n-1} \cdot \frac{p_n - 4}{p_n} \cdot \frac{M}{N} = r_{n-1} \cdot \frac{p_n - 4}{p_n} \cdot \frac{p_n^2}{N}. \tag{5}$$

$$T_{n-1} = r_{n-2} \cdot \frac{p_{n-1} - 4}{p_{n-1}} \cdot \frac{p_{n-1}^2}{N}. \tag{6}$$

We can obtain that

$$T_n = \frac{(p_n - 4)p_n}{p_{n-1}^2} \cdot T_{n-1},$$

where $T_1 = \dfrac{11 \times 7}{70} \cong 1.10$. $(r_{n-1} = \dfrac{1}{70} \cdot N \cdot \dfrac{p_1 - 4}{p_1} \cdot ... \cdot \dfrac{p_{n-1} - 4}{p_{n-1}})$. By

$$\begin{aligned} T_n &= \frac{p_n(p_n-4)}{p_{n-1}^2} \cdot \frac{p_{n-1}(p_{n-1}-4)}{p_{n-2}^2} \cdot \frac{p_{n-2}(p_{n-2}-4)}{p_{n-3}^2} \cdot ... \cdot \frac{p_2(p_2-4)}{p_1^2} \cdot T_1 \\ &= \frac{p_n}{p_{n-1}} \cdot \frac{(p_n-4)}{p_{n-2}} \cdot \frac{(p_{n-1}-4)}{p_{n-3}} \cdot ... \cdot \frac{p_3-4}{p_1} \cdot \frac{p_2-4}{p_1} \cdot T_1 \end{aligned}$$

For the fact that $\dfrac{p_3-4}{p_1} \cdot \dfrac{p_2-4}{p_1} \cdot T_1 > 1$, $\dfrac{p_i - 4}{p_{i-2}} > 1$ $(i = 4, ..., n)$, so $\lim\limits_{n\to\infty} T_n = \infty$, there are infinite twin prime numbers within 10 (For example, 36 pairs in the range of $10^5$).

## *2.7. Distribution of $n^2 + 1$ Prime Numbers*

**Lemma 7.1** $4(mp+10x)^2+1 \equiv 4(mp-10x)^2+1 \pmod{p}$, where $p$ is an odd prime.

*Proof.*

$$4(mp + 10x)^2 + 1 - 4(mp - 10x)^2 - 1 = 16mxp.$$

□

**Lemma 7.2** If $x \pm y \not\equiv 0 \pmod{p}$, then $4(mp + 10x)^2 + 1 \not\equiv 4(mp + 10y)^2 + 1 \pmod{p}$.

*Proof.*

$$4(mp + 10x)^2 + 1 - 4(mp + 10y)^2 - 1 = 80mp(x - y) + 400(x + y)(x - y).$$

Because 400 and $x \pm y$ mod $p$ are not of congruence, then this Lemma is proved.

□

$n^2 + 1$ is a composite number when $n$ is odd; let $n = 2t$ when $n$ is even, $g$ is the units digit of $t$. $n^2 + 1$ is a multiple of 5, when $n^2 + 1 = 4t^2 + 1$, $g = 1, 4, 6, 9$. And $g = 2, 3, 5, 7, 8, 0$ when $4t^2 + 1$ is a prime number.

$(4n^2 + 1, 4n + 3) = 1$, denote the prime numbers with the form $4n + 1$ by $q_i$.

**Theorem 7.1** On the number axis $4t^2+1$, there are two multiples of $q_i$ at every $q_i$ distance.

*Proof.* By Lemma 7.1, Lemma 7.2 know, $4t^2 + 1$ number axis appear multiples of the $q_i$, distributed $4(mq_i)^2 + 1$ into a center of symmetry.

For example, there are two numbers which are multiples of 13 at every distance of 13 on $4(10k + 2)^2 + 1$ number axis, shown in Figure 1.

Therefore, the surplus model of distribution of $n^2+1$ prime numbers on number axis $4(10k + 2)^2 + 1$ is similar with that of distribution of twin prime numbers on number axis $A_1 - B_3$, and the trend is similar too. As a result, there is no maximum value of twin prime number in natural numbers, and there are infinite prime numbers which have the form $n^2 + 1$.

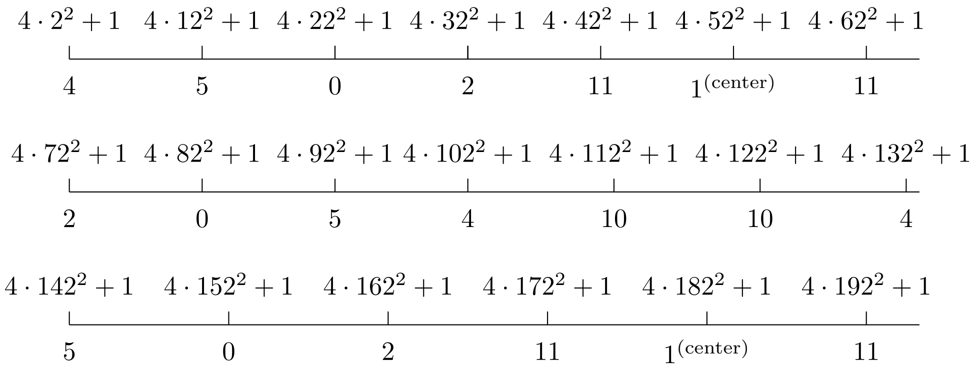


**Figure 1**

Note: Numbers 4, 5, 0, ... shown below the axis are the remainders of $4(10k+2)^2+1$ divided by 13. □

## *2.8. Distribution of Prime Triplet*

If $p_i$, $p_i+2$, $p_i+6$ are all prime numbers, then these three-digits group is called a *prime triplet.*

On number axes group ($A_1-B_3-A_7$ or $A_7-B_9-A_3$), similarly, $r_0=[\frac{4}{105}\cdot N]$; $r_1=[r_0\cdot\frac{p_1-3}{p_1}]$; $r_2=[r_1\cdot\frac{p_2-3}{p_2}]$; ... $r_i=[r_{i-1}\cdot\frac{p_i-3}{p_i}]$.

The method of the surplus numbers $r_{i-1}$ to $r_i$ is called *the residual model of distribution of prime triplet.*

$E_n$ is a prime triplet in the range of $N$, let $M=p_n^2$, by

$$E_n=[r_{n-1}\cdot\frac{p_n-3}{p_n}\cdot\frac{M}{N}]=[r_{n-1}\cdot\frac{p_n-3}{p_n}\cdot\frac{p_n^2}{N}]. \tag{7}$$

$$E_{n-1}=[[r_{n-2}\cdot\frac{p_{n-1}-3}{p_{n-1}}]\frac{p_{n-1}^2}{N}]. \tag{8}$$

We can obtain that

$$E_n=[\frac{(p_n-3)p_n}{p_{n-1}^2}\cdot E_{n-1}],$$

where $E_1=[\frac{4\times 11\times 8}{105}]=3$. ($r_{n-1}=[...[\frac{4}{105}\cdot N]\frac{p_1-3}{p_1}]...]\frac{p_{n-1}-3}{p_{n-1}}]$).

For the fact that $\frac{(p_i-3)p_i}{p_{i-1}^2}>1$, so $\{[\frac{(p_i-3)p_i}{p_{i-1}^2}\cdot S_{i-1}]\}$ $(i=1,2,...,n)$ are ascending series in which all numbers are not less than 3. Therefore, $\lim_{n\to\infty}E_n=\infty$, there are infinite prime triplets in natural numbers.

## *2.9. Primes Distribution of $n^2-n+p_i$ Primes*

**Lemma 9.1** There are only $\frac{p+1}{2}$ residual classes in $n^2-n$ for odd prime $p$.

*Proof.* $n = 1$, $(p \pm 1)(p \pm 0) \equiv 0 \pmod p$;

$n = 2$, $(p \pm 2)(p \pm 1) \equiv 2 \times 1 \pmod p$;

$n = 3$, $(p \pm 3)(p \pm 2) \equiv 3 \times 2 \pmod p$;

...;

$n = \frac{p-1}{2}$, $(p \pm \frac{p-1}{2})(p \pm \frac{p-3}{3}) \equiv \frac{p-1}{2} \times \frac{p-3}{2} \pmod p$;

$n = \frac{p+1}{2}$, $(p \pm \frac{p+1}{2})(p \pm \frac{p-1}{3}) \equiv \frac{p+1}{2} \times \frac{p-1}{2} \pmod p$;

$n = \frac{p+3}{2}$, $(p \pm \frac{p+3}{2})(p \pm \frac{p+1}{3}) \equiv \frac{p+3}{2} \times \frac{p+1}{2} ... \equiv \frac{p-1}{2} \times \frac{p-3}{2} + 2p \equiv \frac{p-1}{2} \times \frac{p-3}{2} \pmod p$;

That is $f(\frac{p+3}{2}) \equiv f(\frac{p-1}{2}) \pmod p$. Let $f(n) = (p \pm n)(p \pm (n-1)) \equiv n(n-1) \pmod p$;

$$\frac{p+1+2t}{2} \times \frac{p+1+2(t-1)}{2} = \frac{p+1-2t+4t}{2} \times \frac{p+1-2(t+1)+4t}{2}$$
$$= \frac{p+1-2t}{2} \times \frac{p+1-2(t+1)}{2} + 2tp \equiv \frac{p+1-2t}{2} \times \frac{p+1-2(t+1)}{2} \pmod p.$$

That is $f(\frac{p+1+2t}{2}) \equiv f(\frac{p+1-2t}{2}) \pmod p$.

Therefore, there are only $1 \times 0$, $2 \times 1$, ... , $\frac{p+1}{2} \cdot \frac{p-1}{2}$ different residual classes in $n^2 - n$ for prime $p$. □

$0 \le n \le N$, $n^2 - n + p_i$ are all prime numbers, the first three numbers must be prime triplet, and $p_i$ must be on $A_1$ (or $A_7$) number axis.

**Lemma 9.2** Let $n^2 - n \equiv r_2 \pmod{p_i}$, $A_1$ number axes $\frac{p_i+1}{2}$ residue class are no longer make proposition holds.

*Proof.* Let $a \in A_1$, $a \equiv r_1 \pmod{p_i}$, $r_1 = 0, 1, ..., p_{i-1}$.

$n^2 - n \equiv r_2 \pmod{p_i}$, then by Lemma 9.1, $r_2 = 0, 2, 6, ..., \frac{p_i+1}{2}$.

If $r_1 + r_2 \equiv 0 \pmod{p_i}$, then $a + r_2 \equiv 0 \pmod{p_i}$, $a + r_2$ is a composite number.

Any number in $r_2$ can make $r_1 + r_2 \equiv 0 \pmod{p_i}$ established, then $a + r_2$ is a composite number.

□

Similarly, $r_0 = [\frac{1}{35} \cdot N]$; surplus numbers (with the quantity of $r_0$) divided by $p_1$ respectively residual $0, 1, 2, ..., p_1 - 1$. $n^2 - n$ divided by $p_1$ have different residue class, by Lemma 9.2, $p_1$ exclude $[r_0 \cdot \frac{p_1+1}{2p_1}]$ numbers, $r_1 = r_0 - [r_0 \frac{p_1+1}{2p_1}] = [r_0 \frac{p_1-1}{2p_1}]$; $r_2$ represents the numbers which do not contain factor $p_2$, $r_2 = [r_1 \cdot \frac{p_2-1}{2p_2}]$; ... $r_i = [r_{i-1} \cdot \frac{p_i-1}{2p_i}]$.

The method of the surplus numbers $r_{i-1}$ to $r_i$ is called *the residual model of distribution of* $n^2 - n + p_i$ *prime numbers.*

$Q_n$ is a proposition holds the number of the $N$-range, let $M = p_n^2$, by

$$Q_n = [r_{n-1} \cdot \frac{p_n - 1}{2p_n} \cdot \frac{M}{N}] = [r_{n-1} \cdot \frac{p_n - 1}{2p_n} \cdot \frac{p_n^2}{N}]. \quad (9)$$

$$Q_{n-1} = [[r_{n-2} \cdot \frac{p_{n-1} - 1}{2p_{n-1}}] \frac{p_{n-1}^2}{N}]. \quad (10)$$

We can get:

$$Q_n = [\frac{(p_n - 1)p_n}{2p_{n-1}^2} \cdot Q_{n-1}],$$

where $Q_1 = [\frac{11}{7}] = 1$. $(r_{n-1} = [...[\frac{1}{35} \cdot N]\frac{p_1 - 1}{2p_1}]...]\frac{p_{n-1} - 1}{2p_{n-1}}])$. and then

$$Q_n = [\frac{(p_n - 1)p_n}{2p_{n-1}^2} \cdot Q_{n-1}] \leq \frac{p_n^2}{35 \cdot 2^n} \prod_{i=1}^{n} \frac{p_i - 1}{p_i}.$$

Based of Bertrand assumption, there is at least one prime between $n$ and $2n$ and at least one prime between $2^{n-1}$ and $2^n$. When $n \geq 12$ $(p_{12} = 53)$,

$$\frac{p_i^2}{2^i} < 1, \ (i \geq 12), \ \frac{p_i - 1}{p_i} < 1, \ (i = 1, 2, ..., n).$$

Thereby, $\lim_{n \to \infty} Q_n = 0$.

There are infinitely primes $p_i$ when $N = 2, 3$; all $n^2 - n + p_i$ are primes when $0 \leq n \leq N$; only some special natural numbers $N$ when $N = p_i - 1$; all $n^2 - n + p_i$ are primes when $0 \leq n \leq N$. The greater $N$ is, the smaller possibility the proposition holds. Therefore, there are only 6 primes $(p_i = 2, 3, 5, 11, 17, 41)$ that make the proposition hold.

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