CSCanada
Studies in Mathematical Sciences
Vol. 7, No. 2, 2013, pp. [79–89]
DOI: 10.3968/j.sms.1923845220130702.2483

ISSN 1923-8444 [Print]
ISSN 1923-8452 [Online]
www.cscanada.net
www.cscanada.org

The Dual Space χ^2 of Double Sequences

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Received: June 13, 2013/ Accepted: September 22, 2013/ Published: November 30, 2013

Abstract: We determine the $\beta(v)-$ dual of the space and establish that the $\alpha-$ and $\gamma-$ duals of the space χ^2 not coincide with the $\beta(v)-$ dual; where $v \in \{p, bp, r\}$.

Key words: Double gai sequences; Double analytic; Double gai; Dual

Subramanian, N., & Misra, U. K. (2013). The Dual Space χ^2 of Double Sequences. *Studies in Mathematical Sciences, 7*(2), 79–89. Available from http://www.cscanada.net/index.php/sms/article/view/j.sms.1923845220130702.2483 DOI: 10.3968/j.sms.1923845220130702.2483

1. INTRODUCTION

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [4]. Later on, they were investigated by Hardy [8], Moricz [12], Moricz and Rhoades [13], Basarir and Solankan [2], Tripathy [20], Colak and Turkmenoglu [6], Turkmenoglu [22], and many others.

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{ (x_{mn}) \in w^2 : sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_p(t) := \left\{ (x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 1 \, for \, some \, l \in \mathbb{C} \right\},$$

$$\mathcal{C}_{0p}(t) := \left\{ (x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1 \right\},$$

$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}; \mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [27,28] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [29] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [30] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [31] and Mursaleen and Edely [32] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x.

More recently, Altay and Basar [33] have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [34] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [35] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a + b)^p \leq a^p + b^p \tag{1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see [1]).

A sequence $x = (x_{mn})$ is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double entire sequence if $|x_{mn}|^{1/m+n} \to 0$ as $m, n \to \infty$. The double entire sequences will be denoted by Γ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m + n)! \, |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{all \, finite \, sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\Im_{ij}$ for all $m,n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a 1 in the $(i,j)^{th}$ place for each $i,j \in \mathbb{N}$.

An FK-space (or a metric space) X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m,n \in \mathbb{N})$ are also continuous.

If X is a sequence space, we give the following definitions:

(i) $X' =$ the continuous dual of X;

(ii) $X^{\alpha} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \, for\, each\, x \in X\right\}$;

(iii) $X^{\beta} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn}\, is\, convegent,\, foreach\, x \in X\right\}$;

(iv) $X^{\gamma} = \left\{a = (a_{mn}) : sup_{mn} \geq 1 \left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, foreachx \in X\right\}$;

(v) $let\, X\, beanFK - space \supset \phi;\, then\, X^{f} = \left\{f(\Im_{mn}) : f \in X'\right\}$;

(vi) $X^{\delta} = \left\{a = (a_{mn}) : sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty,\, foreach\, x \in X\right\}$;

X^{α}, X^{β}, X^{γ} are called $\alpha-(orK\ddot{o}the-Toeplitz)$ dual of X, $\beta-(or\, generalized-K\ddot{o}the-Toeplitz)\, dual\, of\, X, \gamma-\, dual\, of\, X,\, \delta-dual\, of\, X\, respectively.X^{\alpha}$ is defined by Gupta and Kamptan [24]. It is clear that $x^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\alpha} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [36] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_{∞}, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Here w, c, c_0 and ℓ_{∞} denote the classes of all, convergent,null and bounded sclar valued single sequences respectively. The above spaces are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k\geq 1} |\Delta x_k|.$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m,n \in \mathbb{N}$.

Let X be the space of double sequences, converging with respect to some linear convergence rule $v - lim : X \to \Re$. The sum of a double series $\sum_{i,j} x_{ij}$ with respect to this rule is defined by $v - \sum_{i,j} x_{ij} := v - limS_{mn}$. We denote w^2 and Ω are called as the double sequence spaces respectively. Let us define the following sets of double sequences: A sequence $x = (x_{mn}) \in \Omega$ is said to be double analytic of t if

$$sup_{mn} |x_{mn}|^{t_{mn}/m+n} < \infty.$$

The vector space of all prime sense double analytic sequences are usually denoted by $\Lambda^2(t)$.

If $t_{mn} = 1$ then a sequence $x = (x_{mn}) \in \Omega$ is said to be double analytic if

$$sup_{mn} |x_{mn}|^{1/m+n} < \infty.$$

The vector space of all prime sense double analytic sequences are usually denoted by Λ^2. The space Λ^2 is a metric space with the metric

$$d(x,y) = sup_{mn} \left\{ |x_{mn} - y_{mn}|^{1/m+n} : m,n = 1,2,\cdots \right\} \tag{2}$$

for all $x = (x_{mn})$ and $y = (y_{mn})$ in Λ^2, respectively.

A sequence $x = (x_{mn}) \in \Omega$ is called a double entire sequence if

$$p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}/m+n} = 0$$

We denote $\Gamma_p^2(t)$ as the class of prime sense double entire sequences.

$$\Gamma_{bp}^2(t) = \Gamma_p^2(t) \bigcap \Lambda^2(t)$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m,n \in \mathbb{N}$.

In the case $t_{mn} = 1$ for all $m,n \in \aleph$; $\Lambda^2(t)$, $\Gamma_p^2(t)$ and $\Gamma_{bp}^2(t)$ reduce to the sets Λ^2, Γ_p^2 and Γ_{bp}^2, respectively.

In the present paper, we introduce the space χ^2 :

A sequence $x = (x_{mn}) \in \Omega$ is called a double gai sequence if

$$((m+n)! |x_{mn}|)^{1/m+n} \to 0 \text{ as } m,n \to \infty.$$

We denote χ^2 as the class of prime sense double gai sequences. The space χ^2 is a metric space with the metric

$$\tilde{d}(x,y) = sup_{mn} \left\{ ((m+n)! |x_{mn} - y_{mn}|)^{1/m+n} : m,n = 1,2,\cdots \right\} \tag{3}$$

for all $x = (x_{mn})$ and $y = (y_{mn})$ in χ^2, respectively.

2. THE DOUBLE SEQUENCE SPACE χ^2

In this section, we give the some inclusion relations concerning the space χ^2, we establish that the $\alpha-$ and $\gamma-$ duals of a space of double sequences are identical whenever it is solid and determine $\beta(v)-$ dual of the space χ^2 for $v \in \{p, bp, r\}$ which is not coincides with the $\alpha-$ and $\gamma-$ duals of the space χ^2.

The $\alpha-$ dual X^α, $\beta(v)-$ dual $X^{\beta(v)}$ with respect to the $v-$ convergence for $v \in \{p, bp, r\}$ and the $\gamma-$ dual X^γ of a double sequence space X are respectively defined by

(i) $X^\alpha = \left\{ a = (a_{mn}) \in \Omega : \sum_{m,n=1}^{\infty} |a_{mn} x_{mn}| < \infty, \, for\, all\, x \in X \right\}$

(ii) $X^{\beta(v)} = \left\{ a = (a_{mn}) \in \Omega : v - \sum_{m,n=1}^{\infty} a_{mn} x_{mn}\, exists,\, forall\, x \in X \right\}$

(iii) $X^\gamma = \left\{ a = (a_{mn}) \in \Omega : sup_{M,N\in\mathbb{N}} \left| \sum_{m,n=1}^{M,N} a_{mn} x_{mn} \right| < \infty, foreachx \in X \right\}$;

It is easy to see for any two spaces λ, μ of double sequences that $\mu^\alpha \subset \lambda^\alpha$ whenever $\lambda \subset \mu$ and $\lambda^\alpha \subset \lambda^\gamma$. Additionally, it is known that the inclusion $\lambda^\alpha \subset \lambda^{\beta(v)}$ holds while the inclusion $\lambda^{\beta(v)} \subset \lambda^\gamma$ does not hold, since the $v-$ convergence of the sequence of partial sums of a double series does not imply its boundedness.

The space λ of double sequence is said to be solid if and only if

$$\tilde{\lambda} = \{(y_{mn}) \in \Omega : \exists\,(x_{mn}) \in \lambda\, such that\; |y_{mn}| \le |x_{mn}|\; for\, all\, m, n \in \mathbb{N}\} \subset \lambda$$

The space λ of double sequences is also said to be monotone if and only if $m_0\lambda \subset \lambda$ where m_0 is the span of the set of all sequences of zero's and one's and

$$m_0\lambda = \{ax = (a_{mn}x_{mn}) : a \in m_0, x \in \lambda\}.$$

If λ is monotone, then $\lambda^{\alpha} = \lambda^{\beta(v)}$ ([29], p. 36) and λ is monotone whenever λ is solid

Prior to giving the theorem which asserts that the $\alpha-$ and $\gamma-$ duals of a solid space of double sequences are identical, we quote two lemmas which are needed in proving the theorem.

Lemma 1 ([50], Theorem 2, p. 279)

A positive term double series converges to its l.u.b (that is the l.u.b of its partial sums) if it is bounded above. otherwise it diverges to $+\infty$.

Lemma 2 ([49], p. 382)

A double series is absolutely convergent if and only if if the set

$$\left\{\sum_{i,j=1}^{m,n} |x_{ij}| : m, n \in \mathbb{N}\right\}$$

is a bounded set of all real numbers.

3. MAIN RESULTS

3.1. Proposition

χ^2 is solid.

Proof. Let $|x_{mn}| \le |y_{mn}|$ with $y = (y_{mn}) \in \chi^2$.

$$((m+n)!\,|x_{mn}|)^{1/m+n} \le ((m+n)!\,|y_{mn}|)^{1/m+n}.$$

But $((m+n)!\,|y_{mn}|)^{1/m+n} \in \chi^2$, because $y \in \chi^2$. That is,

$$((m+n)!\,|y_{mn}|)^{1/m+n} \to 0\, as\, m, n \to \infty$$

and

$$((m+n)!\,|x_{mn}|)^{1/m+n} \to 0\, as\, m, n \to \infty.$$

Therefore, $x = (x_{mn}) \in \chi^2$. This completes the proof.

3.2. Theorem 1

The $\alpha-$ dual of the space Λ^2 is the space η^2, where

$$\eta^2 = \bigcap_{N \in N - \{1\}} \left\{x = (x_{mn}) \in \Omega : \sum_{mn} |x_{mn}|\, N^{m+n} < \infty\right\}$$

.

Proof. First we show that $\eta^2 \subset \left(\Lambda^2\right)^{\alpha}$. Let $x \in \eta^2$ and $y \in \Lambda^2$. Then we can find a positive integer N such that

$$|y_{mn}|^{1/m+n} < max\left(1, sup_{m,n\geq 1}\left(|y_{mn}|^{1/m+n}\right)\right) < N\, for\, all\, m, n.$$

Hence we may write

$$|\sum_{mn} x_{mn}y_{mn}| \leq \sum_{mn} |x_{mn}y_{mn}| \leq \sum_{mn} |x_{mn}| N^{m+n}$$

Since $x \in \eta^2$, the series on the right side of the above inequality is convergent, whence $x \in \left(\Lambda^2\right)^\alpha$. Hence

$$\eta^2 \subset \left(\Lambda^2\right)^\alpha \tag{4}$$

Now we show that $\left(\Lambda^2\right)^\alpha \subset \eta^2$. For this, let $x \in \left(\Lambda^2\right)^\alpha$, and suppose that $x \notin \Lambda^2$. Then there exists a positive integer $N > 1.$ such that

$$\sum_{mn} |x_{mn}| N^{m+n} = \infty$$

If we define $y_{mn} = N^{m+n}\, Sgn\, x_{mn}\, m, n = 1, 2, \cdots$, then $y \in \Lambda^2$. But, since

$$|\sum_{mn} x_{mn}y_{mn}| = \sum_{mn} |x_{mn}y_{mn}| = \sum_{mn} |x_{mn}| N^{m+n} = \infty$$

we get $x \notin \left(\Lambda^2\right)^\alpha$, which contradicts to the assumption $x \in \left(\Lambda^2\right)^\alpha$. Therefore $x \in \eta^2$

$$\left(\Lambda^2\right)^\alpha \subset \eta^2 \tag{5}$$

From (4) and (5) we are granted $\left(\Lambda^2\right)^\alpha = \eta^2$. This completes the proof.

3.3. Theorem 2

The $\alpha-$ dual of the space χ^2 is the space η^2, where

$$\eta^2 = \bigcap_{N\in N-\{1\}} \left\{x = (x_{mn}) \in \Omega : \sum_{mn} |x_{mn}| N^{m+n} < \infty\right\}$$

.

Proof. We know that $\chi^2 \subset \Lambda^2$. $\Rightarrow$ $\left(\Lambda^2\right)^\alpha \subset \left(\chi^2\right)^\alpha$. But $\left(\Lambda^2\right)^\alpha = \eta^2$, by Theorem 5.2. Therefore

$$\eta^2 \subset \left(\chi^2\right)^\alpha \tag{6}$$

For this, let $x \in \left(\chi^2\right)^\alpha$, and suppose that $x \notin \chi^2$. Then there exists a positive integer $N > 1$ such that $\sum_{mn} |x_{mn}| \frac{1}{(m+n)!} N^{m+n} = \infty$. If we define

$$y_{mn} = \frac{1}{(m+n)!} N^{m+n} Sgn\, x_{mn}\, m, n = 1, 2, \cdots,$$

then $y \in \chi^2$. But, since

$$|\sum_{mn} x_{mn}y_{mn}| = \sum_{mn} |x_{mn}y_{mn}| = \sum_{mn} |x_{mn}| \frac{1}{(m+n)!} N^{m+n} = \infty,$$

we get $x \notin \left(\chi^2\right)^\alpha$, which contradicts to the assumption $x \in \left(\chi^2\right)^\alpha$. Therefore $x \in \eta^2$.

$$\left(\chi^2\right)^\alpha \subset \eta^2 \tag{7}$$

From (6) and (7) we are granted $\left(\chi^2\right)^\alpha = \eta^2$. This completes the proof.

3.4. Theorem 3

If a given double sequence space χ^2 is solid, then the equality $\left(\chi^2\right)^\alpha = \left(\chi^2\right)^\gamma$ holds.

Proof. It is enough show that the inclusion $\left(\chi^2\right)^\gamma \subset \left(\chi^2\right)^\alpha$ holds. Suppose that the sequence space χ^2 is solid and take $y = (y_{mn}) \in \chi^\gamma$. Then,

$$\sup_{i,j \in N} \left| \sum_{m,n=1}^{i,j} x_{mn} y_{mn} \right| < \infty$$

for any $x = (x_{mn}) \in \chi^2$. Now, define the sequence $z = (z_{mn})$ via the sequence $x = (x_{mn}) \in \chi^2$ by

$$((m+n)! z_{mn})^{1/m+n} = ((m+n)! x_{mn})^{1/m+n} \, Sgn \, ((m+n)! x_{mn} y_{mn})^{1/m+n}$$

for all $m, n \in N$. Then $z = (z_{mn}) \in \chi^2$. Since χ^2 is solid and $|z_{mn}| \leq |x_{mn}|$ for all $m, n \in N$. Therefore

$$\begin{aligned} & \sup_{i,j} \sum_{m,n=1}^{i,j} |x_{mn} y_{mn}| \\ = & \sup_{i,j} \sum_{m,n=1}^{i,j} ((m+n)! x_{mn})^{1/m+n} \, Sgn \, \left((m+n)! \, x_{mn} y_{mn}\right)^{1/m+n} \\ = & \sup_{i,j \in N} \left| \sum_{m,n=1}^{i,j} y_{mn} z_{mn} \right| < \infty \end{aligned}$$

This shows that the positive term double series $\sum_{mn} |x_{mn} y_{mn}|$ is bounded which is convergent by Lemma (3). Therefore, Once can see by Lemma 4 that $(x_{mn} y_{mn})_{mn \in N} \in \chi^2$. Since $x \in \chi^2$ is arbitrary, y must be in $\left(\chi^2\right)^\alpha$, (i.e)the inclusion $\left(\chi^2\right)^\gamma \subset \left(\chi^2\right)^\alpha$ holds. Similarly $\left(\chi^2\right)^\alpha \subset \left(\chi^2\right)^\gamma$ holds. This step is easy. Therefore not given to the proof. This completes the proof.

3.5. Theorem 4

If χ^2 is solid then $\left(\chi^2\right)^\alpha = \left(\chi^2\right)^\gamma \neq \left(\chi^2\right)^{\beta(v)}$.

Proof. We observe that the double sequence space χ^2 is solid. This yields to us that the double sequence space χ^2 is monotone which implies the fact that the $\alpha-$duals, $\gamma-$ duals and the $\beta\,(v)-$ duals of the space χ^2 are not identical. This completes the proof.

3.6. Theorem 5

The $\beta\,(v)-$ dual of the space χ^2 is the space Λ^2.

Proof. Let us take any $x \in \Lambda^2$ and $y \in \chi^2$. Consider the inequalities

$$|x_{mn} y_{mn}| \leq |x_{mn}|_{\Lambda^2} + |y_{mn}|_{\chi^2}$$

satisfied for all $m, n \in N$. Therefore, we derive that

$$\sum_{mn} |x_{mn}y_{mn}| \le \sum_{mn} |x_{mn}|_{\Lambda^2} + \sum_{mn} |y_{mn}|_{\chi^2} < \infty,$$

which leads us to the fact that $x \in \left(\chi^2\right)^{\alpha}$, (i.e.) the inclusions

$$\Lambda^2 \subset \left(\chi^2\right)^{\alpha} \subset \left(\chi^2\right)^{\beta(v)} \tag{8}$$

hold.

Conversely, take any $y = (y_{mn}) \in \left(\chi^2\right)^{\beta(v)}$. For establishing the inclusion $\left(\chi^2\right)^{\beta(v)} \subset \Lambda^2$. Let us consider the linear functional f_{pq} and the double sequence $y^{[pq]}$ defined by

$$f_{pq} : \chi^2 \longmapsto \Re$$

$$x = (x_{mn}) \longmapsto f_{pq} := \sum_{m,n=1}^{k=1} x_{mn}y_{mn}$$

and

$$y^{[pq]} = \begin{pmatrix} y_{11}, & y_{13}, & ...y_{1n}, & 0, & ... \\ y_{21}, & y_{23}, & ...y_{2n}, & 0, & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ y_{n1}, & y_{n2}, & ...y_{nn}, & 0, & ... \\ 0, & 0, & ...0, & 0, & ... \end{pmatrix}$$

for every $p, q \in N$. Then, since $y^{[pq]} \in \Lambda^2$, we obtain by Hölders inequality

$$|f_{pq}(x)| \le \sum_{m,n=1}^{k} |x_{mn}y_{mn}| = \sum_{mn} \left|x_{mn}y^{[pq]}\right| \le [d(x,0)]_{\chi^2} \cdot \left[d\left(y^{[pq]},0\right)\right]_{\Lambda^2}$$

for each $x = (x_{mn}) \in \chi^2$ which yields the continuity of the linear functionals f_{pq}. Therefore, we have

$$\|f_{pq}\| \le \left[d\left(y^{[pq]},0\right)\right]_{\Lambda^2}, \quad \text{for each} \quad p, q \in N. \tag{9}$$

Let us consider the sequence $x^{(pq)} = \left\{x_{mn}^{(pq)}\right\}_{m,n\in N}$ to prove the reverse inequality, defined by

$$x_{mn}^{(pq)} = \begin{cases} \dfrac{|y_{mn}|_{\Lambda^2}}{y_{mn}}, & \text{if } y_{mn} \ne o, \quad \text{and} \quad m,n \le p,q, \\ 0, & \text{otherwise} \end{cases}$$

Then, it is clear that $x^{(pq)} \in \chi^2$ and one can see that

$$\left[d\left(x^{(pq)},0\right)\right]_{\chi^2} = \left[d\left(y^{[pq]},0\right)\right]_{\Lambda^2}.$$

This leads us to the consequence for all $p, q \in N$ that

$$\frac{\left|f_{pq}\left(x^{(pq)}\right)\right|}{\left[d\left(x^{(pq)},0\right)\right]_{\chi^2}} = \frac{\left(\sum_{m,n=1}^{k} |y_{mn}|^{1/m+n}\right)_{\Lambda^2}}{\left[d\left(x^{(pq)},0\right)\right]_{\chi^2}} = \left[d\left(y^{[pq]},0\right)\right]_{\Lambda^2}.$$

Hence,

$$\left[d\left(y^{[pq]},0\right)\right]_{\Lambda^2} \leq \|f_{pq}\| \; for\, all\, p,q \in N \tag{10}$$

Therefore, we have (8) and (9) that $\|f_{pq}\| = \left[d\left(y^{[pq]},0\right)\right]_{\Lambda^2}$ $for\, all\, p,q \in N$.

By applying the Banach-Steinhauss Theorem, one can observe by our hypothesis that the sequence (f_{pq}) of linear functionals converges pointwise. Since $\left(\chi^2, |.|_{\chi^2}\right)$ and $(C, |.|)$ are Banach metric spaces, the linear functional defined by

$$f_{st} : \chi^2 \longmapsto \Re$$

$$x = (x_{mn}) \longmapsto f_{st}(x) = lim_{p,q\to\infty} f_{pq}(x) = \sum_{mn} x_{mn} y_{mn}$$

is continuous, and

$$\|f_{st}\| \leq sup_{p,q\in N} \|f_{pq}\|$$
$$= \sup_{p,q\in N} \left[d\left(y^{[pq]},0\right)\right]_{\Lambda^2} < \infty$$

holds. Thus, we have $y \in \Lambda^2$ because of

$$\|f_{st}\| \leq \sup_{p,q\in N} \left[d\left(y^{[pq]},0\right)\right]_{\Lambda^2}$$
$$= \sup_{p,q\in N} \left(\sum_{m,n=1}^{p,q} |y_{mn}|^{m+n}\right)_{\Lambda^2}^{1/m+n}$$
$$= \left(\sum_{mn} |y_{mn}|^{m+n}\right)_{\Lambda^2}^{1/m+n} < \infty$$

That is to say that the inclusion

$$\left(\chi^2\right)^{\beta(v)} \subset \Lambda^2 \tag{11}$$

From (8) and (11) we are granted $\left(\chi^2\right)^{\beta(v)} = \Lambda^2$. This completes the proof.

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