Studies in Mathematical Sciences
Vol. 1, No. 1, 2010, pp. 30-37
www.cscanada.org

ISSN 1923-8444 [Print]
ISSN 1923-8452 [Online]
www.cscanada.net

1-Soliton Solution of the Coupled Nonlinear Klein-Gordon Equations

Ryan Sassaman[1]
Matthew Edwards[2]
Fayequa Majid[3]
Anjan Biswas[1,*]

Abstract: This paper studies the coupled Klein-Gordon equations in (1+1) and (1+2) dimensions. The cubic law of nonlinearity and arbitrary power law nonlinearity are considered in this paper. The 1-soliton solution of the coupled system, for both cases, is obtained. The solitary wave ansatz is used to carry out the integration.

Key Words: Solitons; Integrability; Coupled Klein-Gordon Equations

1. INTRODUCTION

The nonlinear Klein-Gordon equation (NKGE) is an important equation in the area of nonlinear evolution equations (NLEE) [1-10]. NKGE arises in theoretical physics, particularly in the area of relativistic quantum mechanics. There has been various methods that has been applied to carry out the integration of this equation. They are variational iteration method, exponential function method, Adomian decomposition method, G'/G method of integration, semi-inverse variational principle. In this paper, the focus is going to be on the integration of 2-coupled NKGE with cubic nonlinearity. The solitary wave ansatz method will be used to carry out the integration. Finally, the results will be extended to the case of 2-coupled NKGE with arbitrary power law nonlinearity. In both cases, the results will be studied in (1+1) and (1+2) dimensions [3].

2. CUBIC NONLINEARITY

[1]Department of Mathematical Sciences, Center for Research and Education in Optical Sciences and Applications, Delaware State University, Dover, DE, 19901-2277, USA.
[2]Department of Physics, School of Arts and Sciences, Alabama A & M University, Normal, AL-35762, USA.
[3]Department of Mathematics, Alabama A & M University, Normal, AL-35762, USA.
[*]Corresponding author. E-mail address: biswas.anjan@gmail.com.
[†]Received 5 November 2010; accepted 24 November 2010.

In this section, the coupled NKGE will be studied with cubic law of nonlinearity. The study will be split into the following two subsections which are on (1+1) and (1+2) dimensions respectively.

2.1 (1+1) Dimensions

The dimensionless form of the 2-coupled NKGE in (1+1)-dimensions with cubic nonlinearity is given by [3]

$$q_{tt} - k^2 q_{xx} + a_1 q + b_1 q^3 + c_1 q r^2 = 0, \tag{1}$$

$$r_{tt} - k^2 r_{xx} + a_2 r + b_2 r^3 + c_2 q^2 r = 0, \tag{2}$$

where in (1) and (2), the dependent variables q and r are the wave fields while x and t are the independent variables that respectively represent the spatial and temporal variables. This coupled equation was already studied in 2005 where doubly periodic solutions were obtained [3]. In this paper, the search is going to be for the non-topological 1-soliton solution to (1) and (2). Thus, the solitary wave ansatze are taken to be [6]

$$q(x,t) = \frac{A_1}{\cosh^{p_1}\tau}, \tag{3}$$

and

$$r(x,t) = \frac{A_2}{\cosh^{p_2}\tau}, \tag{4}$$

where

$$\tau = B(x - vt), \tag{5}$$

and in (3) and (4), A_1 and A_2 represent the soliton amplitudes while B is the inverse width of the soliton and v is the soliton velocity. The unknown exponents p_1 and p_2 that are to be determined that will be discovered by the balancing method during the soliton solution derivation process. Thus, from (3) and (4)

$$q_{tt} = \frac{p_1^2 v^2 A_1 B^2}{\cosh^{p_1}\tau} - \frac{p_1(p_1+1)v^2 A_1 B^2}{\cosh^{p_1+2}\tau}, \tag{6}$$

$$q_{xx} = \frac{p_1^2 A_1 B^2}{\cosh^{p_1}\tau} - \frac{p_1(p_1+1)A_1 B^2}{\cosh^{p_1+2}\tau}, \tag{7}$$

and

$$r_{tt} = \frac{p_2^2 v^2 A_2 B^2}{\cosh^{p_2}\tau} - \frac{p_2(p_2+1)v^2 A_2 B^2}{\cosh^{p_2+2}\tau}, \tag{8}$$

$$r_{xx} = \frac{p_2^2 A_2 B^2}{\cosh^{p_2}\tau} - \frac{p_1(2+1)A_2 B^2}{\cosh^{p_2+2}\tau}. \tag{9}$$

Substituting (6)-(9) into (1) and (2) respectively gives

$$\frac{p_1^2\left(v^2-k^2\right)A_1 B^2}{\cosh^{p_1}\tau} - \frac{p_1(p_1+1)\left(v^2-k^2\right)A_1 B^2}{\cosh^{p_1+2}\tau} + \frac{a_1 A_1}{\cosh^{p_1}\tau} + \frac{b_1 A_1^3}{\cosh^{3p_1}\tau} + \frac{c_1 A_1 A_2^2}{\cosh^{p_1+2p_2}\tau} = 0, \tag{10}$$

and

$$\frac{p_2^2\left(v^2-k^2\right)A_2B^2}{\cosh^{p_2}\tau}-\frac{p_2(p_2+1)\left(v^2-k^2\right)A_2B^2}{\cosh^{p_2+2}\tau}$$
$$+\frac{a_2A_2}{\cosh^{p_2}\tau}+\frac{b_2A_2^3}{\cosh^{3p_2}\tau}+\frac{c_2A_1^2A_2}{\cosh^{2p_1+p_2}\tau}=0. \tag{11}$$

From (10) equating the exponents $3p_1$ and p_1+2, by the aid of balancing principle, gives [7, 8]

$$3p_1=p_1+2, \tag{12}$$

which yields

$$p_1=1, \tag{13}$$

and similarly, from (11), equating the exponents $3p_2$ and p_2+2 also yields

$$p_2=1. \tag{14}$$

Now, from (10), the linearly independent functions are $1/\cosh^{p_1+j}\tau$ for $j=1,2$ and hence setting their respective coefficients to zero yields

$$B=\sqrt{\frac{-a_1}{v^2-k^2}}, \tag{15}$$

and

$$b_1A_1^2+c_1A_2^2+2a_1=0. \tag{16}$$

Similarly from (11), the linearly independent functions are $1/\cosh^{p_2+j}\tau$ for $j=1,2$ and hence setting their respective coefficients to zero yields [6]

$$B=\sqrt{\frac{-a_2}{v^2-k^2}}, \tag{17}$$

and

$$c_2A_1^2+b_2A_2^2+2a_1=0. \tag{18}$$

From (15) and (18) equating the two values of the soliton width B gives

$$a_1=a_2, \tag{19}$$

and finally solving the coupled system of equations with the soliton amplitudes yield

$$A_1=\sqrt{\frac{2a_1\left(c_1-b_2\right)}{b_1b_2-c_1c_2}}, \tag{20}$$

and

$$A_2=\sqrt{\frac{2a_1\left(b_1-c_2\right)}{c_1c_2-b_1b_2}}. \tag{21}$$

Hence, the 1-soliton solution to (1) and (2) are respectively given by

$$q(x,t)=\frac{A_1}{\cosh[B(x-vt)]}, \tag{22}$$

and

$$r(x,t)=\frac{A_2}{\cosh[B(x-vt)]}, \tag{23}$$

where the amplitudes A_1 and A_2 are respectively given by (20) and (21), while the soliton width B is given by (15) or (17). These introduce the solvability condition given by (19).

2.2 (1+2) Dimensions

The (1+2) dimensional extension of the NKGE with cubic law of nonlinearity is given by

$$q_{tt} - k^2 \left(q_{xx} + q_{yy}\right) + a_1 q + b_1 q^3 + c_1 q r^2 = 0, \tag{24}$$

$$r_{tt} - k^2 \left(r_{xx} + r_{yy}\right) + a_2 r + b_2 r^3 + c_2 q^2 r = 0, \tag{25}$$

where in (24) and (25), the dependent variables q and r are the wave fields. The independent variables x and y are both spatial variables, in this case and t stays as the temporal variables. The solitary wave ansatze are same as given by (3) and (4) respectively with τ in this case being given by

$$\tau = B_1 x + B_2 y - vt. \tag{26}$$

Here, in (26), B_1 and B_2 are the inverse widths of the two solitons in the x and y directions respectively and again v is the soliton velocity. The unknown exponents p_1 and p_2 will again be computed later. Thus, from (3) and (26)

$$q_{tt} = \frac{p_1^2 v^2 A_1}{\cosh^{p_1} \tau} - \frac{p_1(p_1+1) v^2 A_1}{\cosh^{p_1+2} \tau}, \tag{27}$$

$$q_{xx} = \frac{p_1^2 A_1 B_1^2}{\cosh^{p_1} \tau} - \frac{p_1(p_1+1) A_1 B_1^2}{\cosh^{p_1+2} \tau}, \tag{28}$$

$$q_{yy} = \frac{p_1^2 A_1 B_2^2}{\cosh^{p_1} \tau} - \frac{p_1(p_1+1) A_1 B_2^2}{\cosh^{p_1+2} \tau}, \tag{29}$$

and

$$r_{tt} = \frac{p_2^2 v^2 A_2}{\cosh^{p_2} \tau} - \frac{p_2(p_2+1) v^2 A_2}{\cosh^{p_2+2} \tau}, \tag{30}$$

$$r_{xx} = \frac{p_2^2 A_2 B_2^2}{\cosh^{p_2} \tau} - \frac{p_1(2+1) A_2 B_2^2}{\cosh^{p_2+2} \tau}, \tag{31}$$

$$r_{yy} = \frac{p_2^2 A_2 B_2^2}{\cosh^{p_2} \tau} - \frac{p_1(2+1) A_2 B_2^2}{\cosh^{p_2+2} \tau}. \tag{32}$$

Substituting (27)-(32) into (24) and (25) respectively yields

$$\frac{p_1^2 \left(v^2 - k^2 B_1^2 - k^2 B_2^2\right) A_1}{\cosh^{p_1} \tau} - \frac{p_1(p_1+1) \left(v^2 - k^2 B_1^2 - k^2 B_2^2\right) A_1 B^2}{\cosh^{p_1+2} \tau}$$
$$+ \frac{a_1 A_1}{\cosh^{p_1} \tau} + \frac{b_1 A_1^3}{\cosh^{3p_1} \tau} + \frac{c_1 A_1 A_2^2}{\cosh^{p_1+2p_2} \tau} = 0, \tag{33}$$

and

$$\frac{p_2^2 \left(v^2 - k^2 B_1^2 - k^2 b_2^2\right) A_2}{\cosh^{p_1} \tau} - \frac{p_2(p_2+1) \left(v^2 - k^2 B_1^2 - k^2 B_2^2\right) A_2}{\cosh^{p_2+2} \tau}$$

$$+\frac{a_2A_2}{\cosh^{p_2}\tau}+\frac{b_2A_2^3}{\cosh^{3p_2}\tau}+\frac{c_2A_1^2A_2}{\cosh^{2p_1+p_2}\tau}=0. \tag{34}$$

Similarly from (33) and (34), as before the same values of p_1 and p_2 are obtained as in the previous case. Identifying the linearly independent functions in (33) and (34) yields the relations [7, 8]

$$B_1^2+B_2^2=\frac{v^2+a_1}{k^2}, \tag{35}$$

$$B_1^2+B_2^2=\frac{v^2+a_2}{k^2}, \tag{36}$$

along with the same coupled equations for the amplitudes that are given by (16) and (18). Therefore the amplitudes A_1 and A_2 are the same as in (20) and (21) and from (35) and (36), the same constraint condition as in (19) is obtained. Hence, finally, the 1-soliton solution to (24) and (25) is given by

$$q(x,y,t)=\frac{A_1}{\cosh(B_1x+B_2y-vt)}, \tag{37}$$

and

$$r(x,y,t)=\frac{A_2}{\cosh(B_1x+B_2y-vt)}, \tag{38}$$

where the amplitudes A_1 and A_2 are respectively given by (20) and (21), while the soliton widths B_1 and B_2 are given by (35) or (36). These solutions introduce the solvability condition given by (19).

3. POWER LAW NONLINEARITY

In this section, the coupled NKGE will be studied with power law of nonlinearity. The study will be split into the following two subsections which are on (1+1) and (1+2) dimensions respectively.

3.1 (1+1) Dimensions

The dimensionless form of the 2-coupled NKGE in (1+1)-dimensions with cubic nonlinearity is given by [9]

$$q_{tt}-k^2q_{xx}+a_1q+b_1q^{m+n}+c_1q^mr^n=0, \tag{39}$$

$$r_{tt}-k^2r_{xx}+a_2r+b_2r^{m+n}+c_2q^nr^m=0, \tag{40}$$

where in (39) and (40), the exponents m and n are positive numbers. The starting hypothesis is going to be the same as (3) and (4). In this case (10) and (11) respectively reduce to

$$\frac{p_1^2\left(v^2-k^2\right)A_1B^2}{\cosh^{p_1}\tau}-\frac{p_1(p_1+1)\left(v^2-k^2\right)A_1B^2}{\cosh^{p_1+2}\tau}$$
$$+\frac{a_1A_1}{\cosh^{p_1}\tau}+\frac{b_1A_1^{m+n}}{\cosh^{(m+n)p_1}\tau}+\frac{c_1A_1^mA_2^n}{\cosh^{mp_1+np_2}\tau}=0, \tag{41}$$

and

$$\frac{p_2^2\left(v^2-k^2\right)A_2B^2}{\cosh^{p_2}\tau}-\frac{p_2(p_2+1)\left(v^2-k^2\right)A_2B^2}{\cosh^{p_2+2}\tau}$$

$$+\frac{a_2A_2}{\cosh^{p_2}\tau}+\frac{b_2A_2^{m+n}}{\cosh^{(m+n)p_2}\tau}+\frac{c_2A_1^nA_2^m}{\cosh^{np_1+mp_2}\tau}=0. \tag{42}$$

From (41) equating the exponents $(m+n)p_1$ and p_1+2 gives [7, 8]

$$(m+n)p_1=p_1+2, \tag{43}$$

which gives

$$p_1=\frac{2}{m+n-1}, \tag{44}$$

and similarly, from (42), equating the exponents $(m+n)p_2$ and p_2+2 that yields the same value of p_2 as in p_1 seen in (44).

Now, from (42), the linearly independent functions are $1/\cosh^{p_1+j}\tau$ for $j=1,2$ and hence setting their respective coefficients to zero yields

$$B=\frac{m+n-1}{2}\sqrt{\frac{-a_1}{v^2-k^2}}, \tag{45}$$

and

$$2b_1A_1^{m+n-1}+2c_1A_1^{m-1}A_2^n+(m+n-1)a_1=0. \tag{46}$$

Similarly from (42), the linearly independent functions are $1/\cosh^{p_2+j}\tau$ for $j=1,2$ and hence setting their respective coefficients to zero yields

$$B=\frac{m+n-1}{2}\sqrt{\frac{-a_2}{v^2-k^2}}, \tag{47}$$

and

$$2c_2A_1^nA_2^{m-1}+2b_2A_2^{m+n-1}+(m+n-1)a_1=0. \tag{48}$$

From (45) and (47), equating the two values of the soliton width B gives the same constraint condition as (19). The amplitudes of the soliton are obtained by solving the coupled system given by (46) and (48). Hence, the 1-soliton solutions to (39) and (40) are given by

$$q(x,t)=\frac{A_1}{\cosh^{\frac{2}{m+n-1}}[B(x-vt)]}, \tag{49}$$

and

$$r(x,t)=\frac{A_2}{\cosh^{\frac{2}{m+n-1}}[B(x-vt)]}, \tag{50}$$

where the amplitudes A_1 and A_2 are respectively given by the coupled system (46) and (48), while the soliton width B is given by (45) or (47). These lead to the solvability condition given by (19).

3.2 (1+2) Dimensions

The (1+2) dimensional extension of the NKGE with cubic law of nonlinearity is given by

$$q_{tt}-k^2\left(q_{xx}+q_{yy}\right)+a_1q+b_1q^{m+n}+c_1q^mr^n=0, \tag{51}$$

$$r_{tt} - k^2\left(r_{xx} + r_{yy}\right) + a_2 r + b_2 r^{m+n} + c_2 r^m q^n = 0, \tag{52}$$

where in (51) and (52), the dependent variables q and r are the wave fields while x, y and t are the independent variables that respectively represent the spatial and temporal variables. In order to solve (51) and (52) for soliton solution the starting hypothesis is the same as (3) and (4) with τ being given by (26). Substituting (27)-(32) into (51) and (52) respectively yields

$$\begin{aligned}&\frac{p_1^2\left(v^2 - k^2 B_1^2 - k^2 B_2^2\right) A_1}{\cosh^{p_1}\tau} - \frac{p_1(p_1+1)\left(v^2 - k^2 B_1^2 - k^2 B_2^2\right) A_1 B^2}{\cosh^{p_1+2}\tau}\\ &+\frac{a_1 A_1}{\cosh^{p_1}\tau} + \frac{b_1 A_1^{m+n}}{\cosh^{(m+n)p_1}\tau} + \frac{c_1 A_1^m A_2^n}{\cosh^{mp_1+np_2}\tau} = 0,\end{aligned} \tag{53}$$

and

$$\begin{aligned}&\frac{p_2^2\left(v^2 - k^2 B_1^2 - k^2 b_2^2\right) A_2}{\cosh^{p_1}\tau} - \frac{p_2(p_2+1)\left(v^2 - k^2 B_1^2 - k^2 B_2^2\right) A_2}{\cosh^{p_2+2}\tau}\\ &+\frac{a_2 A_2}{\cosh^{p_2}\tau} + \frac{b_2 A_2^{m+n}}{\cosh^{(m+n)p_2}\tau} + \frac{c_2 A_1^n A_2^m}{\cosh^{np_1+mp_2}\tau} = 0.\end{aligned} \tag{54}$$

Equations (53) and (54) yields as before the same values of p_1 and p_2 as in (44). Identifying the linearly independent functions in (53) and (54) yields the relations

$$B_1^2 + B_2^2 = \frac{4v^2 + a_1(m+n-1)^2}{4k^2}, \tag{55}$$

$$B_1^2 + B_2^2 = \frac{4v^2 + a_2(m+n-1)^2}{4k^2}, \tag{56}$$

along with the same coupled equations for the amplitudes that are given by (46) and (48) and the same constraint condition as in (19) is obtained. Hence, finally, the 1-soliton solution to (51) and (52) is given by

$$q(x,y,t) = \frac{A_1}{\cosh^{\frac{2}{m+n-1}}(B_1 x + B_2 y - vt)}, \tag{57}$$

and

$$r(x,y,t) = \frac{A_2}{\cosh^{\frac{2}{m+n-1}}(B_1 x + B_2 y - vt)}, \tag{58}$$

where the amplitudes A_1 and A_2 are respectively given by the coupled system (46) and (48), while the soliton width B is given by (55) or (56). These give the solvability condition given by (19). Additionally, the solitons (57) and (68) introduce the constraint on the exponents m and n that is given by

$$m + n > 1, \tag{59}$$

which must hold for the solitons to exist.

ACKNOWLEDGEMENT

This research work for first (RS) and last (AB) authors was fully supported by NSF-CREST Grant No: HRD-0630388 and this support is genuinely and sincerely appreciated.

REFERENCES

[1] Akhmediev, N., & Ankiewicz, A. (1997). *Solitons, nonlinear pulses and beams*. London: Chapman and Hall Publishers.

[2] Basak, K. C., Ray, P. C., & Bera, R. K. (2009). Solution of non-linear Klein-Gordon equation with a quadratic non-linear term by Adomian decomposition method. *Commun. Nonl. Sci. Numer. Simul., 14*, 718-723.

[3] Liu, C. P. (2005). New doubly periodic solutions for the coupled nonlinear Klein-Gordon equations. *Commun. Theor. Phys., 43*, 13-16.

[4] Crutcher, S. H., Osei, A. J., & Edwards, M. E. (2008). Optical spatial solitons, the power law and the swing effect. *Proceedings of SPIE*, 7056.

[5] Sammelson, R. E., Heidari A., & Tayyari, S. F. (2010). An analytical approach to the Klein-Gordon and Dirac relativistic oscillators in non-commutative space under a constant magnetic field. *Commun. Nonl. Sci. Numer. Simul., 15*, 1368-1371.

[6] Sassaman, R., Heidari, A., Majid, F., Zerrad, E., & Biswas, A. (2010). Topological and non-topological solitons of the generalized Klein-Gordon equation in (1+2) dimensions. *Dynam. Cont. Discr. Impul. Sys. Series A, 17*, 275-286.

[7] Wazwaz, A. M. (2005). The tanh and sine-cosine methods for compact and noncompact solutions of the nonlinear Klein-Gordon equation. *Appl. Math. Comput., 167*, 1179-1195.

[8] Wazwaz, A. M. (2005). Generalized forms of the phi-four equation with compactons, solitons and periodic solutions. *Math. Comput. Simul., 69*, 580-588.

[9] Zhang, J., Gan, Z. H., & Guo, B. L. Stability of standing waves for a class of coupled nonlinear Klein-Gordon equations. (2010). *Acta Math. Appl. Sin., 26*, 427-442.

[10] Zheng, Y., & Lai S. (2009). A study of three types of nonlinear Klein-Gordon equations. *Dynam. Cont. Discr. Impul. Sys. Series B, 16*, 271-279.